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January 2016

# Ordinary Differential Equations - Preliminary Exam

Closed books, closed notes. Show work for full credit. No calculators are allowed.

1. Let $(x(t), y(t))$ be a nontrivial solution of the initial value problem

$$\begin{aligned} \dot{x} &= 2x + 4y \quad , \quad x(0) = x_0 \\ \dot{y} &= 4x + 2y \quad , \quad y(0) = y_0 \end{aligned}$$

   i.e., assume that we have $(x_0, y_0) \neq (0, 0)$. Find all possible values of $\lim_{t\to\infty} y(t)/x(t)$. Which initial conditions $(x_0, y_0)$ lead to which limit?

2. Consider the autonomous linear system $\dot{z} = Az$ of ordinary differential equations, and let $|\cdot|$ denote the Euclidean 2-norm. Consider the two following two statements: If false provide a counterexample, and if true provide a proof.

   (a) Suppose that all the eigenvalues of $A$ have negative real part. Then every solution of $\dot{z} = Az$ satisfies
   $$|z(t)| \leq |z(s)| \qquad \text{for all} \qquad t > s\ .$$

   (b) Suppose that $A$ is symmetric and all the eigenvalues of $A$ have negative real part. Then every solution satisfies
   $$|z(t)| \leq |z(s)| \qquad \text{for all} \qquad t > s\ .$$

3. Consider the autonomous planar ordinary differential equation

$$\begin{aligned} \dot{x} &= -x - 2y^2\ , \\ \dot{y} &= xy - y^3\ . \end{aligned}$$

   (a) Find all equilibrium solutions of this system, and determine whether the Principle of Linearized Stability applies at any of them.

   (b) Show that the function $V(x, y) = x^2 + 2y^2$ is a Lyapunov function for the system, and use it to determine the stability of the origin $(0, 0)$.

4. Consider the autonomous ordinary differential equation

$$\begin{aligned} \dot{x} &= -x\ , \\ \dot{y} &= -y + 2x^2\ , \\ \dot{z} &= z + 2x^2\ . \end{aligned}$$

   (a) Find the general solution of the system.

   (b) Find the stable manifold and the unstable manifold at the origin $(0, 0, 0)$.