Linear Analysis Preliminary Exam August 2019

(1) Suppose $f : X \to X$ is a Lipschitz mapping from a complete metric space to itself. That is, for some $C > 0$ one has $d(f(x), f(y)) \leq Cd(x, y)$ for all $x, y \in X$. For which values of C is f guaranteed to have a fixed point? Prove that this restriction on C is necessary in order for f to have a fixed point, or give a counter-example.

(2) A sequence v_i in a normed space V converges weakly to v if $f(v_i)$ converges to $f(v)$ for every linearly functional in V^*. Prove that the standard basis vectors e_i in ℓ_2 converge weakly to the zero vector.

(3) Let X be a real Hilbert space and let $P : X \to X$ be a bounded, self-adjoint linear operator satisfying $P^2 = P$. Furthermore, let $M = N(I - P)$.

  (a) Show that M is a closed linear subspace of X.

  (b) Show that P is the orthogonal projection of X onto M.

(4) (a) Carefully stating any results that you use, prove that any sufficiently small perturbation ΔA of a bounded linear invertible operator A between two Banach spaces E and E_1 itself has a bounded inverse.

  (b) If $E = E_1$ and $\|A\| < 1$ then prove that $(I - A)^{-1}$ can be represented as an infinite sum.