Linear Analysis Preliminary Exam January 2019

(1) Suppose X and Y are homeomorphic metric spaces. Suppose X is complete. Is it true that Y is complete? Give a proof or counter-example.

(2) State the Weierstrass approximation theorem. Use it to prove that there is a collection of polynomials $\{p_i\}_{i\in\mathbb{N}}$ such that for each $f \in C_2[a,b]$, there are coefficients $a_i \in \mathbb{R}$ such that one has $f = \sum_{i=1}^{\infty} a_i p_i$.

(3) Consider the Banach space l^2 of square-summable complex sequences $\{x_1, x_2, \ldots\}$.

(a) Construct a bounded linear operator $L: \ l^2 \to l^2$ for which

$$\|L\| := \sup_{\{x\in l^2 : \|x\| \neq 0\}} \frac{\|Lx\|}{\|x\|} = 1$$

but where for every non-zero sequence $\|Lx\| < \|x\|$.

(b) Can such an operator L be compact? Either find one or prove it is not possible.

(4) Let X be a normed linear space and X^* its dual space.

(a) State the Hahn-Banach theorem.

(b) Prove that $x \in X$ satisfies $x = 0$ if and only if $x^*(x) = 0$ for all $x^* \in X^*$.

(c) Prove that for any two elements $x, y \in X$ there exists a functional $x^* \in X^*$ such that $x^*(x) \neq x^*(y)$.