Linear Analysis Preliminary Exam August 2018

(1) Let K be the set of all nonempty compact subsets of $\mathbb{R}$. For $A, B \in K$ define

$$d(A,B) = \sup_{x\in A} \inf_{y\in B} |x-y| + \sup_{y\in B} \inf_{x\in A} |x-y| \ .$$

Prove that (K, d) is a metric space. Is this metric space separable?

(2) Let X be a normed linear space and X^* its dual space.

(a) State the Hahn-Banach theorem.

(b) Prove that for any two elements $x, y \in X$ there exists a functional $x^* \in X^*$ such that $x^*(x) \neq x^*(y)$.

(3) Let $K : [0,1]^2 \to \mathbb{R}$ be a continuous function which satisfies $K(t,s) = K(s,t)$ for all $t, s \in [0,1]$, and then consider the operator T acting on functions by the formula

$$(Tf)(t) = \int_0^1 K(t,s) f(s)\, ds \ .$$

(a) Show that $T : L^2[0,1] \to L^2[0,1]$ is linear and continuous.

(b) Find an integral representation for the adjoint $T^* : L^2[0,1] \to L^2[0,1]$.

(4) A subset C of a Hilbert space H is *convex* if for all $x, y \in C$ and $0 \leq t \leq 1$ the linear combination $tx + (1-t)y \in C$. Prove that, given a nonempty, closed, convex subset $C \subseteq H$, for every element $x \in H$ there exists an element $y \in C$ such that

$$\|x - y\| \leq \|x - z\| \quad \text{for all} \quad z \in C \ .$$

In other words, the minimal distance from x to C is attained at a point $y \in C$. Is this point y uniquely determined by x?

Hint: Construct a minimizing sequence $(x_n) \subset C$ and argue that it is Cauchy. For this, use the facts that $\|x_m - x_n\| = \|(x - x_n) - (x - x_m)\|$ and that H is a Hilbert space.