Linear Analysis Preliminary Exam January 2018

(1) Let (X, d) be a complete metric space, and let $(A_n)_{n=1}^{\infty}$ denote a nested sequence of closed subsets $A_n \subset X$, i.e., assume that $A_{n+1} \subset A_n$ for all $n \in \mathbb{N}$.

(a) Give an example in $\mathbb{R}$, equipped with the usual distance $d(x, y) = |x - y|$, of such a sequence for which $\bigcap_{n=1}^{\infty} A_n$ is the empty set.

(b) If A_1 is compact, show that the intersection $\bigcap_{n=1}^{\infty} A_n$ is not empty.

(c) Give an example in $\mathbb{R}$, equipped with the distance $d(x, y) = |x - y|$, for which the intersection in (b) contains at least two points. What additional assumption guarantees that the intersection contains exactly one point?

(2) Let X be a real Hilbert space and let $P : X \to X$ be a continuous, self-adjoint linear operator satisfying $P^2 = P$. Show that there exists a closed linear subspace $M \subset X$ such that P is the orthogonal projection P_M onto M.

(3) Consider the space l_2 of all sequences $x = (x_1, x_2, \ldots)$ with $\sum_{j=1}^{\infty} x_j^2 < \infty$, and equipped with the usual norm. Prove that every continuous linear functional on the space l_2 is of the form

$$f(x) = \sum_{j=1}^{\infty} c_j x_j$$

with an appropriate condition on the numbers c_j, which you should state and verify.

(4) Let $K : [0, 1]^2 \to \mathbb{R}$ be a continuous function with $K(t, s) < 1$ for all $t, s \in [0, 1]$. Show that for each $y \in C[0, 1]$ the integral equation

$$x(t) - \int_0^1 K(t, s)x(s)\, ds = y(t)\ , \qquad \text{for all } t \in [0, 1]$$

has a unique solution $x \in C[0, 1]$. Furthermore, given an example of a kernel function K which violates the assumption "$K(t, s) < 1$ for all $t, s \in [0, 1]$", and a function $y \in C[0, 1]$, for which the above integral equation has infinitely many solutions.