Department of Mathematical Sciences George Mason University

Linear Analysis Preliminary Exam, August 2011

This exam consists of 4 questions.

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(1) Let $X = C^1[0,1]$ denote the vector space of all continuously differentiable real-valued functions defined on the interval $[0,1]$, and let $\|f\|_\infty = \max\{|f(x)| : x \in [0,1]\}$. Furthermore, define

$$\|f\|_A = \|f\|_\infty , \qquad \|f\|_B = \|f'\|_\infty , \qquad \|f\|_C = \|f\|_\infty + \|f'\|_\infty .$$

(a) Which of these three definitions result in a norm on X?

(b) For which definition is the resulting normed linear space complete? Justify your answer.

(2) Let H denote a Hilbert space, and let $U \subset H$ denote a subspace.

(a) Show that $U \subset \left(U^\perp\right)^\perp$.

(b) In part (a), is it possible to replace U by $[U]$ on the left-hand side of the inclusion, where $[U]$ denotes the closure of U? Give a proof or a counterexample.

(c) In part (a), is it possible to replace the inclusion by an equality? Give a proof or a counterexample.

(3) Consider the Hilbert space $H = L^2(0,1)$ equipped with its usual norm $\|f\|_2 = (\int_0^1 |f(x)|^2 dx)^{1/2}$. Furthermore, define the linear operator $T : H \to H$ by $Tf(x) = \int_0^x f(\xi) d\xi$. (You do not have to verify the linearity and the fact that T maps H into H.)

(a) Prove that T is a bounded linear operator.

(b) What can you say about the continuity and/or smoothness properties of the functions in the range of T? Is the operator T onto?

(c) Determine the point spectrum of T, i.e., the set of all eigenvalues of T.

(4) Let H denote a Hilbert space, let $T \in \mathcal{L}(H,H)$ denote a bounded linear operator, and let $T^* \in \mathcal{L}(H,H)$ denote its adjoint.

(a) Using the definition of the adjoint operator and of the orthogonal complement, prove that

$$N(T) = R(T^*)^\perp \quad \text{and} \quad N(T^*) = R(T)^\perp ,$$

where N and R denote the nullspace and range, respectively.

(b) Assume further that T is a normal operator, i.e., that it satisfies $TT^* = T^*T$. Show that then the identity $\|Tx\| = \|T^*x\|$ holds for all $x \in H$. What does this imply for $N(T)$ and $N(T^*)$?
