Linear Analysis Preliminary Exam, August 2011

This exam consists of 4 questions.

1. Let \( X = C^1[0,1] \) denote the vector space of all continuously differentiable real-valued functions defined on the interval \([0,1] \), and let \( \|f\|_\infty = \max\{|f(x)| : x \in [0,1]\} \). Furthermore, define \( \|f\|_A = \|f\|_\infty, \quad \|f\|_B = \|f'\|_\infty, \quad \|f\|_C = \|f\|_\infty + \|f'\|_\infty \).

(a) Which of these three definitions result in a norm on \( X \)?

(b) For which definition is the resulting normed linear space complete? Justify your answer.

2. Let \( H \) denote a Hilbert space, and let \( U \subset H \) denote a subspace.

(a) Show that \( U \subset (U^\perp)^\perp \).

(b) In part (a), is it possible to replace \( U \) by \([U]\) on the left-hand side of the inclusion, where \([U]\) denotes the closure of \( U \)? Give a proof or a counterexample.

(c) In part (a), is it possible to replace the inclusion by an equality? Give a proof or a counterexample.

3. Consider the Hilbert space \( H = L^2(0,1) \) equipped with its usual norm \( \|f\|_2 = (\int_0^1 |f(x)|^2 \, dx)^{1/2} \). Furthermore, define the linear operator \( T : H \to H \) by \( Tf(x) = \int_0^x f(\xi) \, d\xi \). (You do not have to verify the linearity and the fact that \( T \) maps \( H \) into \( H \).)

(a) Prove that \( T \) is a bounded linear operator.

(b) What can you say about the continuity and/or smoothness properties of the functions in the range of \( T \)? Is the operator \( T \) onto?

(c) Determine the point spectrum of \( T \), i.e., the set of all eigenvalues of \( T \).

4. Let \( H \) denote a Hilbert space, let \( T \in \mathcal{L}(H,H) \) denote a bounded linear operator, and let \( T^* \in \mathcal{L}(H,H) \) denote its adjoint.

(a) Using the definition of the adjoint operator and of the orthogonal complement, prove that \( N(T) = R(T^*)^\perp \text{ and } N(T^*) = R(T)^\perp \).

(b) Assume further that \( T \) is a normal operator, i.e., that it satisfies \( TT^* = T^*T \). Show that then the identity \( \|Tx\| = \|T^*x\| \) holds for all \( x \in H \). What does this imply for \( N(T) \) and \( N(T^*) \)?