Topology Preliminary Exam August 21, 2018

Note: Any space referred to in this exam is both nonempty and Hausdorff.

1. For each part, give an example of $A \subset \mathbb{R}^2$ satisfying the given condition, with a brief justification:

   (a) A is closed but not compact.

   (b) A is connected but not locally connected.

   (c) A contains points p and q for which there is no homeomorphism $h : A \to A$ with $h(p) = q$.

2. Let (X, d) be an uncountable separable metric space. Show that the set $B = \{p \in X : p \text{ is isolated in } X\}$ is countable.

3. A topological space is *zero-dimensional* (with respect to the small inductive dimension) if it has a basis consisting of sets which are both open and closed.

   (a) Show that $\mathbb{Q}$ is zero-dimensional by exhibiting an appropriate basis (don't provide a proof).

   (b) Prove that $\mathbb{Q}$ does not have a one-point compactification.

   (c) Prove that every zero-dimensional space has a Stone-Čech compactification. (Hint: use the continuity of carefully-chosen characteristic functions, and quote a theorem.)

4. Suppose X is a topological space, $(Y, <)$ is an ordered topological space, and that $f : X \to Y$ and $g : X \to Y$ are continuous. Prove that the set $C = \{x \in X : f(x) \leq g(x)\}$ is a closed subset of X.

5. Let $i : \mathbb{R} \to \mathbb{R}_\ell$ denote the identity function from $\mathbb{R}$ to $\mathbb{R}_\ell$, where $\mathbb{R}_\ell$ denotes the Sorgenfrey line (that is, the real numbers with the lower-limit topology). Prove that i is not continuous at any point $x \in \mathbb{R}$.