Linear Analysis Preliminary Exam January 20th 2017

TIME ALLOWED: 2 HOURS

(1) A function $f : [0, T] \to \mathbb{R}$ is defined as *regulated* if both the left and right limits exist for all $t \in [0, T]$ (defining the left limit at 0 to be $f(0)$ and the right limit at T to be $f(T)$), and is *right-continuous* if $\lim_{s \to t^+} f(s) = f(t)$ for all $t \in [0, T]$. Prove that

$$\|f\|_{[0,T]} = \sup\{|f(t)| : 0 \leq t \leq T\}$$

is indeed a norm on the set of all right-continuous regulated functions on $[0, T]$ and that this space is a Banach space.

(2) State and prove the Nested Sphere Theorem for metric spaces. Use it to prove Baire's Theorem.

(3) (a) Prove that the space of continuous linear operators mapping a normed linear space to a complete normed linear space with the operator norm forms a normed linear space.

(b) Prove this space of operators is always complete.

(4) (a) Let U and V be closed subspaces of a Hilbert space with corresponding orthogonal projections P_U and P_V respectively. Show that for any orthogonal projection P, we have $P^2 = P$.

(b) Consider two operators built using P_U and P_V as follows:

$$A = P_U - P_V, \quad B = I - P_U - P_V.$$

Show that $AB + BA = 0$

(c) With the same notation as above, show that $A^2 + B^2 = I$.

(d) Describe briefly the role of orthogonal projections in the proof of the Hilbert-Schmidt theorem for completely continuous (compact) self-adjoint operators.