Prelim in Analysis

1. Define in general the conjugate space (also known as the dual space) to a normed linear space E and prove it is complete.

2. Let (X, d) denote a complete metric space X with metric d, and let $f : X \to X$ be a mapping from X into itself. For $n \in \boldsymbol{N}$, the n-th iterate of f is defined by

$$f^n(x) = \underbrace{f(f(\cdots f(}_{n \text{ times}}x)\cdots)) \,,$$

   i.e., we have $f^2(x) = f(f(x))$, $f^3(x) = f(f(f(x)))$, etc.

   (a) Give an example of a complete metric space X and a continuous map $f : X \to X$ such that f has a unique fixed point, but f^2 does not have a unique fixed point.

   (b) Now assume that $f : X \to X$ is a uniform contraction, i.e., assume there exists a constant c with $0 \leq c < 1$ such that $d(f(x), f(y)) \leq c \cdot d(x, y)$ for all $x, y \in X$. Show that in this case, for every $n \in \boldsymbol{N}$ the iterate f^n has a unique fixed point.

3. Prove that the linear functional T defined on $C[0, 1]$ by

$$Tx(t) = \int_0^{1/2} x(t)\, dt - \int_{1/2}^1 x(t)\, dt$$

   is a bounded linear functional on $C[0, 1]$. Find its norm.

4. Let A be the operator in l_2 (with real scalars) such that

$$Ax = A(x_1, x_2, x_3, \ldots) = (0, x_1, \frac{x_2}{2}, \frac{x_3}{3}, \ldots).$$

   Show that A on l_2 is completely continuous (alias compact). Find the adjoint of the operator A. Find the spectrum of A, including but not limited to all the eigenvalues of A.