Show all work. No work = No points. This test is closed book and closed notes. Use only the methods discussed in this class. You have **120 minutes** to complete this test. Good Luck!

Complete four of the following six problems.

(1) (10 points) Let (X, d) denote a not necessarily complete metric space, and let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ denote two Cauchy sequences in X. Show that then the sequence $(z_n)_{n\in\mathbb{N}}$ defined by $z_n = d(x_n, y_n)$ converges.

(2) (10 points) Let (X, d) denote a complete metric space, for every natural number $n \in \mathbb{N}$ let $A_n \subset X$ be open and dense, and let $A = \cap_{n\in\mathbb{N}} A_n$ denote the intersection of these sets.

(a) Is A always dense? Justify your answer or give a counterexample.

(b) Is A always open? Justify your answer or give a counterexample.

(3) (10 points) Let X denote an inner product space over the field K.

(a) For $K = \mathbb{R}$, prove that two elements $x, y \in X$ are orthogonal if and only if they satisfy the Pythagorean theorem, i.e., if and only if the identity $\|x+y\|^2 = \|x\|^2+\|y\|^2$ holds.

(b) Does the result from (a) remain true for $K = \mathbb{C}$? Give a proof or a counterexample.

(4) (10 points) Consider the space $X = C[0, 4]$ equipped with the maximum norm, and define

$$f(x) = x(0) + \int_1^2 x(t)\,dt - \int_3^4 x(t)\,dt \quad \text{for} \quad x \in X\ .$$

Show that then $f \in X^*$ and determine its norm $\|f\|_{X^*}$.

(5) (10 points) Let X be a Banach space and let $A \in \mathcal{L}(X, X)$ be arbitrary.

(a) Show that the series $T = \sum_{k=0}^{\infty} \frac{1}{k!} A^k \in \mathcal{L}(X, X)$ converges.

(b) Express the adjoint T^* in terms of $A^* \in \mathcal{L}(X^*, X^*)$.

(6) (10 points) Consider the Banach space $X = C[0, 1]$ of continuous real-valued functions on the interval $[0, 1]$, equipped with the maximum norm. Furthermore, let $a \in X$ denote the function defined by $a(t) = 2t$ for $0 \le t \le 1/2$ and $a(t) = 1$ for $1/2 \le t \le 1$. Finally, define the operator $T : X \to X$ via

$$(Tx)(t) = a(t)x(t) \quad \text{for all} \quad t \in [0, 1]\ .$$

(a) Determine the resolvent set of T.

(b) Determine the point spectrum of T.