Preliminary Exam, Linear Analysis, August 2009

On all problems, state any theorems that you use.

1. Let (X, d) be a complete metric space. A mapping $T : X \to X$ is called a *contraction* if there exists a constant $\lambda, 0 \leq \lambda < 1$ such that

$$d(Tx, Ty) \leq \lambda d(x, y), x, y \in X.$$

T is called a *contractive map* if it satisfies

$$d(Tx, Ty) < d(x, y)$$

for $x \neq y$.
(a) Prove that if T is a contraction, then T has a fixed point, i.e. there exists $x \in X$ such that $Tx = x$.
(b) Show by an example that a contractive map need not have a fixed point. (Hint: Consider $x + \frac{1}{x}$ on an appropriate interval.)

2. Let H be a Hilbert space.
(a) If P is a bounded linear operator such that $x \in H$ can be written uniquely as $x = x_0 + x_1$, where $x_0 \in Range(P)$, and $x_1 \in Range(P)^{\perp}$, then show that $Range(P)$ is a closed subspace.
(b) Let U and V be closed subspaces of a Hilbert space X, and let P_U and P_V be orthogonal projections of U and V. Show that $U \subset V$ if and only if $P_U = P_V P_U$.

3. Let X be a separable normed linear space. If E is any closed subspace of X, show that there is a sequence of norm one functionals (f_n) in X^* such that $d(x, E) = \sup_n |f_n(x)|$ for all $x \in X$. Conclude that $E = \cap_{n=1}^{\infty} \ker f_n$. Make sure to state any theorem that you use.

   (Hint: Given (x_n) dense in X, use the Hahn-Banach theorem to choose f_n so that $f_n = 0$ on E and $f_n(x_n) = d(x_n, E)$.)

4. Let N denote the natural numbers, and let $H = l^2(N)$, the Hilbert space of square summable sequences on N. Show that the identity operator on H is not compact.