1. Let \((X,d)\) be a complete metric space. A mapping \(T : X \to X\) is called a contraction if there exists a constant \(\lambda, 0 \leq \lambda < 1\) such that
\[
d(Tx, Ty) \leq \lambda d(x, y), \quad x, y \in X.
\]

\(T\) is called a contractive map if it satisfies
\[
d(Tx, Ty) < d(x, y)
\]
for \(x \neq y\).
(a) Prove that if \(T\) is a contraction, then \(T\) has a fixed point, i.e. there exists \(x \in X\) such that \(Tx = x\).
(b) Show by an example that a contractive map need not have a fixed point. (Hint: Consider \(x + \frac{1}{x}\) on an appropriate interval.)

2. Let \(H\) be a Hilbert space.
   (a) If \(P\) is a bounded linear operator such that \(x \in H\) can be written uniquely as \(x = x_0 + x_1\), where \(x_0 \in \text{Range}(P)\), and \(x_1 \in \text{Range}(P)^\perp\), then show that \(\text{Range}(P)\) is a closed subspace.
   (b) Let \(U\) and \(V\) be closed subspaces of a Hilbert space \(X\), and let \(P_U\) and \(P_V\) be orthogonal projections of \(U\) and \(V\). Show that \(U \subset V\) if and only if \(P_U = P_V P_U\).

3. Let \(X\) be a separable normed linear space. If \(E\) is any closed subspace of \(X\), show that there is a sequence of norm one functionals \((f_n)\) in \(X^*\) such that \(d(x, E) = \sup_n |f_n(x)|\) for all \(x \in X\). Conclude that \(E = \bigcap_{n=1}^{\infty} \ker f_n\). Make sure to state any theorem that you use.
   (Hint: Given \((x_n)\) dense in \(X\), use the Hahn-Banach theorem to choose \(f_n\) so that \(f_n = 0\) on \(E\) and \(f_n(x_n) = d(x_n, E)\).)

4. Let \(N\) denote the natural numbers, and let \(H = l^2(N)\), the Hilbert space of square summable sequences on \(N\). Show that the identity operator on \(H\) is not compact.