Department of Mathematical Sciences George Mason University

Algebra Preliminary Exam, January 2019

This exam consists of 5 questions.

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(1) Let $(\mathbb{Q},+)$ denote the additive group of the rational numbers. Show that $(\mathbb{Q},+)$ has a subgroup that is not finitely generated. Is every finitely generated subgroup of $(\mathbb{Q},+)$ cyclic?

(2) Show that a group of order 117 has a normal subgroup.

(3) Let R be a commutative ring.

1. Show that every maximal ideal is prime.
2. If further R is a PID, show that every nonzero prime ideal is maximal.

(4) Let R be the ring given by

$$R = \left\{ \begin{pmatrix} n & 3m \\ m & n \end{pmatrix} : n, m \in \mathbb{Z} \right\}.$$

Show that R is a commutative ring. Show further that an element in R is a unit if and only if n^2 equals either $3m^2 - 1$ or $3m^2 + 1$.

(5) Let α be an irrational real root of $x^2 + bx + c$ where $b, c \in \mathbb{Q}$.

1. Show that $\mathbb{Q}[\alpha] = \mathbb{Q}(\alpha)$, that is, the smallest subring of $\mathbb{R}$ that contains $\mathbb{Q}$ and α, is a subfield of $\mathbb{R}$.
2. Find explicitly the inverse of $b + \alpha$ as an element in $\mathbb{Q}[\alpha]$.
