Topology Preliminary Exam - January 2014

The symbol $\mathbb{R}$ denotes the real numbers, $\mathbb{Q}$ denotes the rationals and $\mathbb{N}$ denotes the natural numbers.

1. Recall that if X and Y are topological spaces, the projection function $\pi_X : X \times Y \to X$ is defined by $\pi_X((x, y)) = x$.

   (a) Prove that π_X must be an open mapping. This means that the image of an open set in $X \times Y$ under π_X is open in X.

   (b) Show that π_X need not be a closed mapping. This means there are X, Y and a closed set $K \subset X \times Y$ whose image under π_X is not closed in X. Hint: You may want to consider the set $K = \{(x, \frac{1}{x}) : x \in \mathbb{R} - \{0\}\} \subset \mathbb{R} \times \mathbb{R}$.

2. Endow both $\mathbb{N} \times \{0, 1\}$ and $\{0, 1\} \times \mathbb{N}$ with the lexicographic order topologies. One of these two spaces is discrete and the other is not. Decide which is which and show why.

3. Let $W = \{[x, x+1] : x \in \mathbb{R}\}$, and suppose $S \subset W$ such that if $s, t \in S$ then $s \cap t = \emptyset$. Show that $\bigcup S$ is a closed set in $\mathbb{R}$.

4. Let $B_1 = \{(a, b) : a, b \in R \text{ and } a < b\}$, and let $B_2 = \{[r, b) : r \in \mathbb{Q} \text{ and } r < b\}$. $B = B_1 \cup B_2$ is a basis for a topology on $\mathbb{R}$(you do not need to show this). Show that the topology determined by B is neither the Sorgenfrey line topology ($\mathbb{R}_L$) nor the usual metric topology on $\mathbb{R}$.

5. Let K be a collection of compact sets in a topological space X which satisfies: If $\{k_1, k_2, \ldots, k_n\}$ is any non empty finite sub-collection of K, then $\cap_{i=1}^{n} k_i \neq \emptyset$. Show that $\bigcap K \neq \emptyset$.