Topology prelim
January, 2013

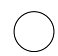
The space of real numbers is denoted $\mathbb{R}$. All metric spaces have the metric topology, products of topological spaces have the Tychonoff product topology, and subspaces of topological spaces have the subspace topology.

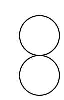
1. Let (X, d) be a metric space. Prove that X is normal.

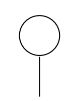
2. Suppose that for each $\lambda \in \Lambda$, the space X_λ is regular. Prove that the product space $\prod_{\lambda \in \Lambda} X_\lambda$ is regular.

3. Suppose that X is a Lindelöf space and $f\colon X \to Y$ is a continuous surjection. Prove that Y is Lindelöf.

4. Give a brief but precise reason that no two of the following subspaces of the Euclidean plane $\mathbb{R}^2$ are homeomorphic.

   (a) The circle $X = \{(x, y) : x^2 + y^2 = 1\}$. 

   (b) The figure 8 $Y = \{(x, y) : x^2 + y^2 = 1\} \cup \{(x, y) : x^2 + (y-2)^2 = 1\}$. 

   (c) The balloon $Z = \{(x, y) : x^2 + (y-2)^2 = 1\} \cup \{(0, y) : 0 \le y \le 1\}$. 

5. Suppose that X is a compact metric space and $\{H_k : k = 1, 2, \cdots\}$ are closed connected subsets of X such that $H_k \supseteq H_{k+1}$ for $k = 1, 2, \cdots$. Prove that $\cap_{k=1}^{\infty} H_k$ is connected.

6. Prove that the space $\mathbb{P}$ of irrational numbers is not the union of countably many compact subsets.