Topology Preliminary Exam
January, 2012

Throughout this exam, $\mathbb{R}$ denotes the set of reals with the usual topology, and subsets of topological spaces are assumed to have the subspace topology.

1. Suppose (X, d) is a metric space. For $K \subseteq X$ and $p \in X$, define $d(p, K)$ to be $\inf\{d(p, x) : x \in K\}$.

   (a) Suppose that K is a compact subset of X and $p \in X$. Prove that there is a $k \in K$ such that $d(p, k) = d(p, K)$.

   (b) Show by example that the result in part (a) can fail if K is only assumed to be a closed subset of X.

2. Prove that a countable complete metric space has an isolated point, that is, a point p such that $\{p\}$ is open.

3. Let X be a compact Hausdorff space. Prove that every infinite subset of X has a limit point.

4. Let S^1 denote the circle $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. For each of the following spaces X, give a reason that S^1 is not homeomorphic to X.

   (a) X is the closed interval $[0, 1]$ in $\mathbb{R}$.

   (b) X is the 2-sphere $\{(x, y, .z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ in $\mathbb{R}^3$.

   (c) X is the union $S^1 \cup \{(x, y) \in \mathbb{R}^2 : (x - 2)^2 + y^2 = 1\}$ of two circles in $\mathbb{R}^2$ which intersect at a single point.

5. Let $\beta\mathbb{N}$ denote the Stone-Čech compactification of the space $\mathbb{N}$ of natural numberswith the discrete topology and let E be the set of even natural numbers. Show that $Cl_{\beta\mathbb{N}}E$ is an open subset of $\beta\mathbb{N}$.