Throughout this exam $\mathbb{R}$ denotes the set of real numbers with the metric $d$ where $d(x, y) = |x - y|$, $\mathbb{P}$ denotes the set of irrational numbers, and $\mathbb{C}$ denotes the Cantor Set. Metric spaces are assumed to have the metric topology, and subsets of topological spaces are assumed to have the subspace topology. Also, $\mathbb{R}_L$ denotes the Sorgenfrey Line (with topology generated by half open intervals), and the symbol $S(\Omega)$ refers to the space of countable ordinals with the natural order topology.

This exam consists of 6 questions.

1. Prove that any bounded and non-empty open set in $\mathbb{R}^2$ has infinitely many boundary points.

2. Let $(X, T)$ be a Hausdorff topological space and let $(Y, T^*)$ be a normal topological space. Prove that the product $X \times Y$, with the product topology, is a Hausdorff topological space.

3. Prove that $\mathbb{R}_L$ does not have a countable basis.

4. Suppose that $A$ is an infinite subset of $S(\Omega)$ Prove that $A$ has a limit point in $S(\Omega)$.

5. If $Y$ is a connected and dense subset in $(X, T)$, prove that $(X, T)$ is a connected topological space.

6. Let $(X, <_x)$ and $(Y, <_Y)$ be well ordered sets. Prove that $X \times Y$ with the lexicographical order is a well ordered set.