# Topology Preliminary Exam

This exam consists of 6 questions. The following notation will be used throughout this exam: $\mathbb{R}$ denotes the set of real numbers and it and all of its subspaces have the usual topology, induced by the metric $d$ where $d(x, y) = |x - y|$.

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**(1)** Suppose that $(X, d)$ is a metric space. Prove that if $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ are bounded uniformly continuous functions, then $fg : X \to \mathbb{R}$ is uniformly continuous, where $(fg)(x) = f(x)g(x)$.

**(2)** Let $(X, d)$ be a metric space. Prove that there exists a subset $S$ of $X$ such that

(i) if $p, q \in S$ with $p \neq q$, then $d(p, q) \geq 1$ and

(ii) if $p \in X$, then there exists a $q \in S$ such that $d(p, q) < 1$.

**(3)** Let $X$ be a connected Tychonov space. Prove that if $|X| > 1$, then $|X| \geq \mathfrak{c}$, where $|X|$ denotes the cardinality of $X$ and $\mathfrak{c} = |\mathbb{R}|$.

**(4)** Suppose that $X$ and $Y$ are regular topological spaces. Prove that $X \times Y$ (with the product topology) is regular.

**(5)** Prove that every subspace of $\mathbb{R}$ is separable.

**(6)** Let $X$ be a compact Hausdorff space. Prove that every infinite subset of $X$ has an accumulation point.

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