Topology Preliminary Exam

This exam consists of 6 questions. The following notation will be used throughout this exam: $\mathbb{R}$ denotes the set of real numbers and it and all of its subspaces have the usual topology, induced by the metric d where $d(x, y) = |x - y|$.

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(1) Prove that if X and Y are Hausdorff spaces, then the product $X \times Y$ with the product topology is also Hausdorff.

(2) Suppose that X is a Hausdorff space and (a_n) is a sequence in X which converges to a and to b. Prove that $a = b$.

(3) Prove that if $f\colon \mathbb{R} \to \mathbb{R}$ is a (not-necessarily continuous) function such that $f(x) > 0$ for all $x \in \mathbb{R}$, then there exists an $\epsilon > 0$ such that $Cl_{\mathbb{R}}\{x \in \mathbb{R} : f(x) > \epsilon\}$ contains a non-empty open interval.

(4) Prove that every compact Hausdorff space is regular.

(5) Let X be a Hausdorff space whose cardinality is at most $\mathfrak{c}$, the cardinality of $\mathbb{R}$. Prove that X has at most $\mathfrak{c}$ closed separable subsets.

(6) Give examples of: (a) a metric space which is not second countable, (b) a first countable Hausdorff space which is not metrizable, and (c) a non first countable Hausdorff space.
