TOPOLOGY PRELIMINARY EXAM
FALL 2014

Problem 1

Let X be a countable, complete metric space. Prove that there is a point $p \in X$ such that $\{p\}$ is an open subset.

Problem 2

Let X, Y be path-connected topological spaces, prove that $X \times Y$ is path-connected.

Problem 3

Prove that a topological space X is Hausdorff if and only if its diagonal $\Delta(X) = \{(x, x)|x \in X\} \subset X \times X$ is closed.

Problem 4

Let X be a locally compact topological space. State what it means for a topological space Y to be a compactification of X. Show that $Y \setminus X$ is a compact topological space.

Problem 5

A topological space X is said to be homogeneous if for any two points $p, q \in X$ there is an automorphism $a \in Aut(X)$ with $a(p) = q$. Decide which of the following topological spaces are homogeneous, and prove your answers.

(1) S^1

(2) $[0, 1]^2$

Problem 6

Let $\mathbb{R}^{\mathbb{N}}$ denote the set of functions $f : \mathbb{N} \to \mathbb{R}$ with the box topology. Let $B \subset \mathbb{R}^{\mathbb{N}}$ be the subset of functions with bounded range. Show that B is non-empty, open and closed, and $\mathbb{R}^{\mathbb{N}} \setminus B$ is nonempty. Is $\mathbb{R}^{\mathbb{N}}$ connected?