Department of Mathematical Sciences George Mason University

Topology Preliminary Exam: January 10, 2020

This exam consists of 6 questions, all of equal weight. You must do *only* 5 questions. DO NOT submit solutions to all 6 problems.

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(1) Let X be a topological space and give $X \times X$ the product topology. Prove that X is Hausdorff if and only if the diagonal $\Delta(X) = \{(x, x) \mid x \in X\}$ is closed in $X \times X$.

(2) Let X and Y be path-connected topological spaces. Prove that $\pi_1(X \times Y)$ is isomorphic to $\pi_1(X) \times \pi_1(Y)$.

(3) Let X be a topological space and Y a set. Let $f : X \to Y$ be a surjective function.

(a) Define a set $V \subset Y$ to be open iff $f^{-1}(V)$ is open in X. Prove that this defines a topology on Y.

(b) Prove that the topology on Y from Part (a) is the finest topology on Y making f continuous.

(c) What is the coarsest topology on Y making f continuous?

(d) With the topology on Y from Part (a), prove that Y is connected if X is connected.

(4) Prove that if $f : X \to Y$ is a proper continuous function between locally compact Hausdorff spaces, then f is a closed mapping.

(5) Prove that every compact metric space is complete and bounded.

(6) Let X be a set, and equip it with the cofinite topology (finite complement topology). Prove that X is metrizable if and only if X is finite.
