Topology Preliminary Exam

This exam consists of 6 questions. The following notation will be used throughout this exam: $\mathbb{R}$ denotes the set of real numbers and it and all of its subspaces have the usual topology, induced by the metric d where $d(x, y) = |x - y|$. Metric spaces are assumed to have the metric topology, and subsets of topological spaces are assumed to have the subspace topology.

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(1) For each of the following pairs of spaces, find a topological property which distingu ishes them, that is, find a topological property which one of the spaces has but the other d oes not.

(a) The open interval $(0, 1)$ and the closed interval $[0, 1]$ in $\mathbb{R}$.

(b) The closed interval $[0, 1]$ in $\mathbb{R}$ and the circle $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ in $\mathbb{R}^2$.

(2) Prove that if (X, d) is a metric space, then X is normal.

(3) Let X and Y be connected topological spaces. Prove that the product $X \times Y$, with the usual product topology, is connected.

(4) Let $\omega = \{0, 1, 2, \cdots\}$ be the set of non-negative integers and $X = [0, 1]^\omega$ have the box topology, that is, a base for the topology is all sets of the form $\prod_{k=0}^{\infty} U_k$ where U_k is open in $[0, 1]$. Let $S = \{(x_k)_{k=0}^{\infty} : \lim_{k\to\infty} x_k = 0\}$. Prove that S is both open and closed in X.

(5) Prove that a compact Hausdorff space is regular.

(6) Suppose that X is a topological space, D is a dense subset of X, and Y is a Hausdorff space. Let $f: X \to Y$ and $g: X \to Y$ be continuous functions such that $f(x) = g(x)$ for every $x \in D$. Prove that $f(x) = g(x)$ for every $x \in X$.
