Topology Preliminary Exam

This exam consists of 6 questions. The following notation will be used throughout this exam: $\mathbb{R}$ denotes the set of real numbers and $\mathbf{I}$ denotes the closed interval $[0,1]$. The "usual" topology on each of these will be as subspaces of $\mathbb{R}$ with the metric d where $d(x,y) = |x-y|$. Also S_Ω denotes the well ordered ordinal space which is uncountable but for which all proper initial segments are countable. Assume all spaces are Hausdorff unless otherwise indicated.

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(1) Prove that any connected open subset of $\mathbb{R}^n$ is path connected.

(2) Let $\{0,1\}$ and $\{0,1,2,3\}$ be considered as discrete topological spaces. Prove that $\{0,1\}^\omega$ and $\{0,1,2,3\}^\omega$ are homeomorphic to each other.

(3) Prove that S_Ω is not metrizable (so no metric on the set S_Ω produces the same topology as the order topology).

(4) Suppose that X is a connected normal space having at least two points. Prove that the cardinality of X is at least the cardinality of $\mathbb{R}$.

(5) Let $[0,1] \times [0,1]$ be endowed with the dictionary order topology and let π_1 and π_2 denote the projection functions to the first and second coordinates. Suppose $f : \mathbf{I} \to [0,1] \times [0,1]$ is continuous. Prove that $\pi_1(f(\mathbf{I}))$ is a constant map.

(6) Prove that a compact Hausdorff space must be normal.
