Department of Mathematical Sciences George Mason University

Topology Preliminary Exam

Throughout this exam $\mathbb{R}$ denotes the set of real numbers with the metric d where $d(x,y) = |x-y|$. Metric spaces are assumed to have the metric topology, and subsets of topological spaces are assumed to have the subspace topology.

This exam consists of 5 questions.

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1. Prove that the circle $\{(x,y) \in \mathbb{R}^2 : x^2+y^2 = 1\}$ is not homeomorphic to the disk $\{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq 1\}$.
2. Let (X,d) be a metric space which is not separable. Prove that X has an uncountable closed, discrete subset.
3. Suppose that X is a countably compact topological space, that is, a topological space such that every open cover with countably many elements has a finite subcover. Prove that every continuous function $f\colon X \to \mathbb{R}$ is bounded.
4. Let X be a Tychonoff space such that $|X| < \mathfrak{c} = 2^{\aleph_0}$. Prove that X has a base $\mathcal{B}$ such that each $B \in \mathcal{B}$ is both open and closed in X.
5. Suppose that X and Y are arc-connected metric spaces. Prove that the product $X \times Y$ is also arc-connected. (Recall that a space X is arc-connected if whenever $p, q \in X$, there is a continuous function $f\colon [0,1] \to X$ such that $f(0) = p$, $f(1) = q$.)
