Department of Mathematical Sciences
George Mason University

August 2018

Ordinary Differential Equations - Preliminary Exam

Closed books, closed notes. Show work for full credit. No calculators are allowed.

1. Let A be a 4×4 matrix. For the following column vectors $v_1 = (1, 2, 3, -4)^t, v_2 = (2, 3, 1, 2)^t, v_3 = (-2, 1, 2, 3)^t, v_4 = (5, 1, -2, 4)^t$, assume that $Av_1 = -4v_1$, $Av_2 = 3v_2$, $Av_3 = 2v_4$, $Av_4 = -2v_3$ (pay close attention to the sub-scripts). (Notation: the superscript t is just the notation for a column vector.)

   (a) For the differential equation $\dot{x} = Ax$, what is the stability of the equilibrium at the origin?

   (b) For the differential equation $\dot{x} = Ax$, what is the maximal stable invariant set? What is the maximal unstable invariant set? Is there another invariant set besides the equilibrium at $(0, 0)$? Specify the precise behavior of solutions within these invariant sets.

   (c) For the differential equation $\dot{x} = Ax + f(x)$, where $f \in C^1$, $f(0) = 0$, $Df(0) = 0$, what is the behavior of solutions in a neighborhood of the origin? Describe the invariant sets and how they relate to the invariant sets in part (b). What is the behavior of solutions within these invariant sets?

2. For the equation $\ddot{x} + x - \dot{x}(1 - x^2 - 2\dot{x}^2) = 0$, noting that x is scalar here, do the following.

   (a) Write the equation as a system in x and y by setting $\dot{x} = y$. Find all equilibria for the system.

   (b) Show that the annulus $1/2 < \sqrt{x^2 + y^2} < 1$ is invariant.

   (c) Show that the annulus $1/2 < \sqrt{x^2 + y^2} < 1$ contains a closed orbit. Justify your answer, stating any theorem that you use.

3. (a) What does it mean for a system of differential equations to be a Hamiltonian? Is the Hamiltonian function unique? What does conservation of energy for a Hamiltonian system mean?

   (b) Find a Hamiltonian for the system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x + x^3 \,. \end{aligned}$$

   (c) Find all equilibrium points and discuss their stability.

   (d) Sketch a phase portrait for this system.

4. Consider the scalar differential equation

$$\dot{x} = rx + x^3 - x^5.$$

   (a) As a function of the parameter r, compute all fixed points of the equation.

   (b) Identify all bifurcation points and classify the bifurcations that occur.