Department of Mathematical Sciences
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Ordinary Differential Equations Preliminary Exam - January 2013

1. Consider the matrix
$$\mathbf{A} = \begin{bmatrix} -1 & -3 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

   (a) Determine a basis of generalized eigenvectors suitable for determining the Jordan form of **A**.

   (b) Use part (a) to calculate $e^{\mathbf{A}t}$.

2. Consider the system
$$\begin{aligned} \dot{x} &= x^3 - 2xy \\ \dot{y} &= -y + x^2 \end{aligned}$$

   (a) Find the first few terms in the power series expansion for the stable and center manifolds of the origin.

   (b) Determine the equation for the flow on the center manifold.

3. (a) Define a C^1 dynamical system on a domain $\mathcal{D} \subseteq \mathbb{R}^n$.

   (b) Suppose $\phi : \mathbb{R} \times \mathcal{D} \to \mathcal{D}$ is a C^1 dynamical system where $\mathcal{D}$ is an open subset of $\mathbb{R}^n$. Let $x_0 \in \mathcal{D}$. Define $y : \mathbb{R} \to \mathcal{D}$ and $f : \mathcal{D} \to \mathcal{D}$ by
$$\begin{aligned} y(t) &: = \phi(t, x_0) \\ f(x) &: = \frac{\partial \phi}{\partial t}(0, x) \end{aligned}$$
   Show that $y(t)$ defined above is a solution to the IVP
$$\dot{y} = f(y), \quad y(0) = x_0$$

4. The equation of motion of a particle is given by
$$\ddot{x} = \begin{cases} -x + c \cdot \operatorname{sgn}(x), & |x| > c \\ 0, & |x| \le c \end{cases}$$
   where $c > 0$.

   (a) Determine all equilibria.

   (b) Sketch a phase portrait for the motion.

   (c) Conjecture on the nature of the non-equilibrium solutions. Try to prove your conjecture.