Department of Mathematical Sciences
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# Ordinary Differential Equations Preliminary Exam - August 2013

Instructions: This exam consists of four problems. You are to do all four of the problems. Closed books, closed notes. State any theorems that you use.

1. Consider the system in $\mathbb{R}^n$ of the form
$$\dot{x} = Ax + h(x)$$
where

   (i) $A$ is an $n$ by $n$ matrix such that all the eigenvalues of $A$ have real part less than zero.

   (ii) $h : \mathbb{R}^n \to \mathbb{R}^n$ satisfies $h(0) = 0$, and
$$\lim_{x \to 0} \frac{|h(x)|}{|x|} = 0.$$

   Do the following

   (a) State a theorem that shows that the equilibrium $x = 0$ is asymptotically stable for the equation $\dot{x} = Ax$.

   (b) Show that the equilibrium $x = 0$ of $\dot{x} = Ax + h(x)$ is asymptotically stable.

2. Consider the system
$$\begin{aligned} \dot{x} &= -x \\ \dot{y} &= 2y + x^2 \end{aligned}$$
Let $\phi(t, x_0, y_0)$ denote the solution of this system with initial condition $(x_0, y_0)$.

   (a) Determine the nature of the equilibrium at the origin. Justify your result by referring to appropriate theorems.

   (b) Define the set $U = \{(x_0, y_0) \in \mathbb{R}^2 : \lim_{t \to -\infty} \phi(t, x_0, y_0) = (0, 0)\}$. Determine the set $U$ for the above system. Justify your answer.

   (c) Define the set $S = \{(x_0, y_0) \in \mathbb{R}^2 : \lim_{t \to \infty} \phi(t, x_0, y_0) = (0, 0)\}$. Determine the set $S$ for the above system. Justify your answer.

3. This problem concerns the system

$$\begin{aligned}\dot{x} &= -x^3 + \lambda x \\ \dot{y} &= -y\end{aligned}$$

(a) Let $\lambda = -1$. Determine all equilibria and classify their stability and type (where type means saddle, node, focus, center, center-focus, or other.) Justify your answer by recourse to an appropriate theorem. Plot a phase portrait.

(b) Let $\lambda = 1$. Determine all equilibria and classify their stability and type. Justify your answer by recourse to an appropriate theorem. Plot a phase portrait.

(c) For all $\lambda$ values, determine the equilibria as a function of $\lambda$, classify their stability and type. For which $\lambda$ values is the phase portrait similar to the portrait in (a), for which $\lambda$ values is it similar to the portrait in (b), and for which $\lambda$ values is it different from either phase portrait?

4. This problem concerns the system

$$\begin{aligned}\dot{x} &= x - y - x^3 \\ \dot{y} &= x + y - y^3\end{aligned}$$

(a) Show how to convert the system to polar coordinates

$$\begin{aligned}\dot{r} &= r\left(1 - r^2\,\frac{3 + \cos 4\theta}{4}\right) \\ \dot{\theta} &= 1 - r^2\,\frac{\sin 4\theta}{4}\end{aligned}$$

Hint: Use the trigonometric identities:

$$\cos^2\theta = \frac{1 + \cos 2\theta}{2}; \quad \sin^2\theta = \frac{1 - \cos 2\theta}{2}$$

(b) Show that $r = 0$ is the only equilibrium.

(c) Show that for $r < 2$, all solutions are moving counterclockwise.

(d) Find the maximum radius $r_1 < 2$ such that all solutions are crossing outward across $r_1$. Find the minimum radius $r_2 < 2$ such that all solutions are crossing inward across $r_2$.

(e) Prove that there is a periodic orbit somewhere in the annulus $r_1 \leq r \leq r_2$.