Ordinary Differential Equations - Preliminary Exam Aug 17, 2009

1. Consider the matrix
$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 3 \end{bmatrix}$$

   (a) Use the Jordan canonical form to compute e^{At}.

   (b) Use part (a) to calculate the solution to the IVP $\dot{\mathbf{x}} = A\mathbf{x}$, $\mathbf{x}(t_0) = \mathbf{x}_0$, for arbitrary $t_0 \in \mathbb{R}$ and $\mathbf{x}_0 \in \mathbb{R}^3$.

2. Consider the nonlinear system
$$\begin{aligned} \dot{x}_1 &= x_1 + e^{x_2} \\ \dot{x}_2 &= -x_2 \end{aligned}$$

   (a) Calculate all equilibria and classify them.

   (b) Determine the stable and unstable manifolds of the equilibria. For each equilibrium, either determine a simple equation or an appropriate approximation for the equation of each manifold.

3. Suppose for a continuous function $\mathbf{f}(t)$ the IVP $\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{f}(t)$, $\mathbf{x}(t_0) = \mathbf{x}_0$ where $A = \begin{bmatrix} -5 & 2 \\ -4 & 1 \end{bmatrix}$ has at least one solution $\mathbf{y} = \boldsymbol{\phi}(t)$ that satisfies $\sup\{|\boldsymbol{\phi}(t)| : t_0 \le t < \infty\} < \infty$.

   (a) Show that all solutions satisfy this boundedness condition.

   (b) State and prove a generalization of this result to an n-dimensional version of the IVP.

4. Let $E \subset \mathbb{R}^n$ is open and let $\mathbf{x}_0 \in E$. Suppose $\mathbf{f} : E \to \mathbb{R}^n$, with $\mathbf{f} \in C^1(E)$, and that $[0, \beta)$ is the right maximal interval of existence of the solution $\mathbf{x} = \boldsymbol{\phi}(t)$ to the IVP $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, $\mathbf{x}(0) = \mathbf{x}_0$.

   (a) Prove that if $\mathbf{y} = \lim_{t \to \beta^-} \boldsymbol{\phi}(t)$ exists, and $\mathbf{y} \in E$, then $\beta = \infty$.

   (b) Prove that $\mathbf{f}(\mathbf{y}) = 0$.

   (c) Prove that $\mathbf{x}(t) \equiv \mathbf{y}$ is a solution to the IVP.