Ordinary Differential Equations - Preliminary Exam  Aug 17, 2009

1. Consider the matrix

\[
A = \begin{bmatrix}
1 & 0 & 0 \\
1 & 2 & 0 \\
1 & 2 & 3 \\
\end{bmatrix}
\]

(a) Use the Jordan canonical form to compute \( e^{At} \).
(b) Use part (a) to calculate the solution to the IVP \( \dot{x} = Ax, \ x(t_0) = x_0 \), for arbitrary \( t_0 \in \mathbb{R} \) and \( x_0 \in \mathbb{R}^3 \).

2. Consider the nonlinear system

\[
\begin{align*}
\dot{x}_1 &= x_1 + e^{x_2} \\
\dot{x}_2 &= -x_2
\end{align*}
\]

(a) Calculate all equilibria and classify them.
(b) Determine the stable and unstable manifolds of the equilibria. For each equilibrium, either determine a simple equation or an appropriate approximation for the equation of each manifold.

3. Suppose for a continuous function \( f(t) \) the IVP \( \dot{x} = Ax + f(t), \ x(t_0) = x_0 \) where \( A = \begin{bmatrix} -5 & 2 \\ -4 & 1 \end{bmatrix} \) has at least one solution \( y = \phi(t) \) that satisfies \( \sup \{|\phi(t)| : t_0 \leq t < \infty \} < \infty \).

(a) Show that all solutions satisfy this boundedness condition.
(b) State and prove a generalization of this result to an \( n \)-dimensional version of the IVP.

4. Let \( E \subset \mathbb{R}^n \) is open and let \( x_0 \in E \). Suppose \( f : E \rightarrow \mathbb{R}^n \), with \( f \in C^1(E) \), and that \([0, \beta)\) is the right maximal interval of existence of the solution \( x = \phi(t) \) to the IVP \( \dot{x} = f(x), \ x(0) = x_0 \).

(a) Prove that if \( y = \lim_{t \to \beta^-} \phi(t) \) exists, and \( y \in E \), then \( \beta = \infty \).
(b) Prove that \( f(y) = 0 \).
(c) Prove that \( x(t) \equiv y \) is a solution to the IVP.