Department of Mathematical Sciences
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August 18, 2014.

Ordinary Differential Equations - Preliminary Exam.

Closed books, closed notes. Show work for full credit. No calculators allowed.

1. Consider the linear system $\mathbf{x}' = A\mathbf{x}$ where A is given by

$$A = \begin{bmatrix} 0 & -2 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

(a) Solve the initial value problem with the initial condition $\mathbf{x}(0) = \mathbf{c}$.
(b) Determine the stable and unstable subspaces.
(c) Sketch the phase portrait.

2. Consider the system

$$\begin{cases} \dot{x} = 2 - 8x^2 - 2y^2 \\ \dot{y} = 6xy \end{cases}$$

(a) Find all the equilibrium points.
(b) Determine the type of stability of these equilibrium points.
(c) Sketch the phase portrait.

3. Suppose $f := R^n \to R^n$ is continuously differentiable. Let $\phi(t, y)$ denote the solution to the initial value problem

$$\dot{x} = f(x) \qquad x(0) = y$$

Assume that x_0 is an asymptotically stable equilibrium of the system $\dot{x} = f(x)$. Show that the set

$$D = \left\{ y \in R^n : \lim_{t \to \infty} \phi(t, y) = x_0 \right\}$$

(a) is positively invariant.
(b) contains the point x_0.
(c) is open.
(d) is connected. [Hint: An open set S in R^n is connected if and only if for any two points $a, b \in S$, there is a path between a and b lying entirely on S.]

4. Consider a system with the total energy given by

$$H(x, y) = \frac{1}{2}y^2 + U(x), \text{ with } U(x) = \frac{1}{2}x^2 - \frac{a}{4}x^4.$$

(a) Write the corresponding dynamical system first in Hamiltonian and then in Newtonian form.
(b) Prove that the total energy of the system remains constant along the trajectories of this dynamical system.
(c) Classify all critical points and sketch the phase portrait of the system.
(d) Find the corresponding orthogonal system and classify it as Hamiltonian, Newtonian or gradient type.