Ordinary Differential Equations Preliminary Exam

This exam consists of 4 questions.

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(1) Suppose that A is an $n \times n$-matrix such that $A^2 = I$.

(a) Find an explicit formula for e^{tA}.

(b) What can you say about the stability properties of the origin of the linear system $\dot{x} = Ax$? Are there nontrivial stable and/or unstable subspaces? Give a general answer, and then illustrate your answer using the matrix

$$A = \begin{bmatrix} 2 & -3 & 4 & -4 \\ 1 & -2 & 4 & -4 \\ 0 & 0 & 2 & -3 \\ 0 & 0 & 1 & -2 \end{bmatrix}.$$

(2) Assume that A and P are two by two matrices such that P is invertible, and $B = P^{-1}AP$, where

$$B = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}$$

Let $x, y \in R^2$.

(a) Find a periodic orbit for the differential equation $\dot{x} = Bx$.

(b) Find a periodic orbit for the differential equation $\dot{y} = Ay$.

turn!

(3) For the equation

$$\ddot{x} - x + x^3 = 0,$$

(a) Write the equation as a first order system of differential equations.

(b) Find a Hamiltonian for the system.

(c) Find all fixed points and their stability type.

(d) Plot the phase portrait.

(4) Consider the planar system

$$\begin{aligned}
\dot{x} &= 2x - y - \frac{2x^3 + 2xy^2 - xy}{\sqrt{x^2+y^2}} \\
\dot{y} &= 2y + x - \frac{2x^2y + 2y^3 + x^2}{\sqrt{x^2+y^2}}
\end{aligned}$$

which in polar coordinates is given by

$$\begin{aligned}
\dot{r} &= 2r(1-r) \\
\dot{\theta} &= 2\sin^2(\theta/2)\ .
\end{aligned}$$

(This does not have to be verified.)

(a) Find all equilibrium points of this system and determine their stability properties. Are the equilibria asymptotically stable?

(b) Sketch the phase portrait of the system, and find all stable and unstable manifolds of the equilibria.

(c) Is the above system a gradient system? Justify your answer.
