Ordinary Differential Equations Preliminary Exam

This exam consists of 4 questions.

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(1) Suppose that A is an $n \times n$-matrix.

(a) If $A^2 = 0$, find an explicit formula for e^{tA}.

(b) Now assume that $A^2 = -I$, and then find an explicit formula for e^{tA}.

(c) Determine the stability of the origin of the linear system $\dot{x} = Ax$ if the coefficient matrix satisfies $A^2 = 0$. What can you say if $A^2 = -I$?

(2) For the system:

$$\dot{x} = x - y - \left(x^2 + \frac{3}{2}y^2\right) x$$
$$\dot{y} = x + y - \left(x^2 + \frac{1}{2}y^2\right) y,$$

(a) Show that when the system is transformed to polar coordinates,

$$\dot{r} = r(1 - r^2) + r^3 \sin^2(\theta) \left(\sin^2(\theta) - \frac{1}{2}\right).$$

(b) For the annulus $A = \{(x, y) : 1/2 < x^2 + y^2 < 2\}$, assuming that there are no equilibria in A, show that the system has a periodic orbit in A.

(3) (a) For the system

$$\dot{x} = x^2 - (x^3 + y^3 + z^3)x$$
$$\dot{y} = y^2 - (x^3 + y^3 + z^3)y$$
$$\dot{z} = z^2 - (x^3 + y^3 + z^3)z,$$

show that the set $C = \{(x, y, z) : x^2 + y^2 + z^2 = 1\}$ is an invariant set.

(b) Let $x \in \mathbb{R}^n$, and let $g : \mathbb{R}^n \to \mathbb{R}$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ be smooth functions. Define the set $S = \{x : g(x) = 0\}$. Find a condition such that S is an invariant set for the differential equation $\dot{x} = f(x)$.

(4) Consider the autonomous system

$$\begin{aligned} \dot{x} &= -y + f(x, y) \\ \dot{y} &= \sin x \,, \end{aligned}$$

where $f : \mathbb{R}^2 \to \mathbb{R}$ is a smooth function.

(a) Show that for $f(x, y) \equiv 0$ the system is Hamiltonian, determine its Hamiltonian $H(x, y)$, and sketch the phase portrait.

(b) For $f(x, y) \equiv 0$, determine the stability properties of all equilibria.

(c) Show that if f satisfies $xf(x, y) \leq 0$ for all $x, y \in \mathbb{R}$, then the equilibrium $(0, 0)$ is stable. (Hint: Can you use the function H from (a)?)
