Ordinary Differential Equations Preliminary Exam

This exam consists of 4 questions.

(1) Suppose that $A$ is an $n \times n$-matrix.
   (a) If $A^2 = 0$, find an explicit formula for $e^{tA}$.
   (b) Now assume that $A^2 = -I$, and then find an explicit formula for $e^{tA}$.
   (c) Determine the stability of the origin of the linear system $\dot{x} = Ax$ if the coefficient matrix satisfies $A^2 = 0$. What can you say if $A^2 = -I$?

(2) For the system:
   \[
   \begin{align*}
   \dot{x} &= x - y - \left( x^2 + \frac{3}{2} y^2 \right) x \\
   \dot{y} &= x + y - \left( x^2 + \frac{1}{2} y^2 \right) y,
   \end{align*}
   \]
   (a) Show that when the system is transformed to polar coordinates,
      \[
      \dot{r} = r(1 - r^2) + r^3 \sin^2(\theta) \left( \sin^2(\theta) - \frac{1}{2} \right),
      \]
   (b) For the annulus $A = \{ (x, y) : 1/2 < x^2 + y^2 < 2 \}$, assuming that there are no equilibria in $A$, show that the system has a periodic orbit in $A$.

(3) (a) For the system
   \[
   \begin{align*}
   \dot{x} &= x^2 - (x^3 + y^3 + z^3)x \\
   \dot{y} &= y^2 - (x^3 + y^3 + z^3)y \\
   \dot{z} &= z^2 - (x^3 + y^3 + z^3)z,
   \end{align*}
   \]
   show that the set $C = \{ (x, y, z) : x^2 + y^2 + z^2 = 1 \}$ is an invariant set.
   (b) Let $x \in \mathbb{R}^n$, and let $g : \mathbb{R}^n \to \mathbb{R}$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ be smooth functions. Define the set $S = \{ x : g(x) = 0 \}$. Find a condition such that $S$ is an invariant set for the differential equation $\dot{x} = f(x)$.

(4) Consider the autonomous system
   \[
   \begin{align*}
   \dot{x} &= -y + f(x, y) \\
   \dot{y} &= \sin x,
   \end{align*}
   \]
   where $f : \mathbb{R}^2 \to \mathbb{R}$ is a smooth function.
   (a) Show that for $f(x, y) \equiv 0$ the system is Hamiltonian, determine its Hamiltonian $H(x, y)$, and sketch the phase portrait.
   (b) For $f(x, y) \equiv 0$, determine the stability properties of all equilibria.
   (c) Show that if $f$ satisfies $xf(x, y) \leq 0$ for all $x, y \in \mathbb{R}$, then the equilibrium $(0, 0)$ is stable. (Hint: Can you use the function $H$ from (a)?)