# Ordinary Differential Equations Preliminary Exam

This exam consists of 4 questions.

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**(1)** Consider the parameter-dependent planar inhomogeneous linear system

$$\dot{x} = A_\alpha x + b_\alpha \ ,$$

where

$$A_\alpha = \begin{bmatrix} 1 & \alpha \\ 4 & 1 \end{bmatrix} \qquad \text{and} \qquad b_\alpha = \begin{bmatrix} 1-\alpha \\ 3 \end{bmatrix} \ ,$$

as well as the associated homogeneous system $\dot{x} = A_\alpha x$.

(a) Determine the stability of the origin for the homogeneous system as a function of the real parameter $\alpha$.

(b) For each value of $\alpha$, find all equilibrium solutions of the inhomogeneous system.

(c) Sketch the phase portraits of the inhomogeneous system for both $\alpha = +1$ and $\alpha = -1$.

**(2)** Consider the second-order differential equation

$$\ddot{x} = \sin x \cos x \ .$$

(a) Write the differential equation as a first order system of two ordinary differential equations, and show that the resulting system is Hamiltonian.

(b) Sketch the phase portrait of the planar system from part (a).

(c) For the planar system from part (a), find the global stable and unstable manifolds of the equilibrium at the origin. Justify your answers.

**(3)** Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a continuously differentiable function with $f(x_0) = 0$, and consider the differential equation

$$\dot{x} = f(x) \ .$$

Moreover, let $V : \mathbb{R}^2 \to \mathbb{R}$ be a nonnegative and differentiable function such that $V(x_0) = 0$, and $V(x) > 0$ for $x \neq x_0$.

(a) Assume that $V$ is decreasing on solutions of the above differential equation, i.e., assume that for every solution $x(t)$ we have $\frac{d}{dt}(V(x(t))) \leq 0$. Show that then the equilibrium $x_0$ is stable.

(b) Use part (a) to show that the equilibrium at the origin for the following system is stable:

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_2 + x_1 x_2 \\ \dot{x}_2 &= x_1 - x_2 - x_1^2 - x_2^3 \ . \end{aligned}$$

(Hint: Try a radially symmetric function $V$.)

**(4)** Consider the system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x + y\left(1 - x^2 - 2y^2\right) \ . \end{aligned}$$

Let $r$ represent the distance of a point to the origin.

(a) Find all equilibria for this system.

(b) Show that $\dot{r} > 0$ for $r = \frac{1}{2}$, and $\dot{r} < 0$ for $r = \frac{3}{2}$.
(Hint: $\frac{d}{dt}(r^2) = \frac{d}{dt}(x^2 + y^2)$. Use this to write $\dot{r}$ in terms of $x$ and $y$.)

(c) Show that the system has a periodic orbit in the annulus $\left\{\frac{1}{2} < r < \frac{3}{2}\right\}$. State any theorem that you use.

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