Department of Mathematical Sciences
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Ordinary Differential Equations - Preliminary Exam.

Closed books, closed notes. Show work for full credit. No calculators allowed.

1. Consider a linear system $\mathbf{x}' = A\mathbf{x}$ where A is given by

$$A = \begin{bmatrix} -1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

   (a) Find the stable, unstable and center subspaces for this system
   (b) Sketch the phase portrait
   (c) Let λ be an eigenvalue of any $n \times n$ matrix A and E be the corresponding generalized eigenspace. Prove that $AE \subset E$.

2. Determine the nature of the critical point at the origin for the following system for all possible values of a:

$$\begin{cases} \dot{x} = -y + ax(x^2 + y^2) \\ \dot{y} = x + ay(x^2 + y^2) \end{cases}$$

   Compare with the behavior of the linearized system. Does this contradict Hartman-Grobman Theorem? Which invariant manifolds exist at the origin? Describe phase portrait as precisely as possible.line

3. Consider the system

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x + x^3 \end{cases}$$

   (a) Find all critical points and determine their geometric nature: i.e., center, focus, saddle, node, and their stability; i.e., stable, asymptotically stable, or unstable.
   (b) Determine the equations for all heteroclinic orbits and sketch them in the phase plane. Be sure to indicate the direction of flow.

4. Determine the critical points and the bifurcation values for the ODE

$$\dot{x} = x\left(x - r^2\right)$$

   Sketch the phase portraits for various values of the parameter r and sketch the bifurcation diagram. Be sure to label all plots appropriately.