Department of Mathematical Sciences
George Mason University January, 2012

Ordinary Differential Equations - Preliminary Exam.

Closed books, closed notes. Show work for full credit. No calculators allowed.

(1) For the ordinary differential equation system

$$\begin{aligned}\dot{x} &= -y + x(1 - x^2 - y^2)\\ \dot{y} &= x + y(1 - x^2 - y^2),\end{aligned}$$

answer the following questions.

(a) Find the linearization of the system at the equilibrium at the origin. What does the principle of linearized stability say about the stability of this equilibrium?

(b) Convert the system to polar coordinates.

(c) Find all equilibria and periodic orbits for the system and describe their stability.

(d) Plot the phase portrait of the system.

(2) For the ordinary differential equation system

$$\begin{aligned}\dot{x} &= -x + y^2\\ \dot{y} &= x - 2y + y^2,\end{aligned}$$

answer the following questions.

(a) Determine all equilibria and classify their stability and type (where type means saddle, node, focus, center, center-focus, or other).

(b) Show that $U = \{(x, y) : y = x, x > 0\}$ is the unstable manifold for one of the equilibria.

(c) Determine a linear approximation of the stable manifold for the equilibrium in part (b).

(d) Prove that $\lim_{t\to\infty} |x(t) - y(t)| = 0$ for any orbit which does not lie on the line $\{(x, y) : y = x\}$.

(3) Consider the differential equation

$$\ddot{x} = F(x), x(0) = x_0, \dot{x}(0) = y_0.$$

(a) Assume that c is a simple zero of F. Let

$$V(x) = -\int_{x_0}^{x} F(s)\, ds.$$

If V has a local minimum at c, determine the stability and type of the equilibrium at the point $(x_0, y_0) = (c, 0)$. (Type of equilibrium means saddle, node, focus, center, center-focus, or other). Use the energy function $E(x) = \dot{x}^2/2 + V(x)$ to explain your answer. What is the stability and type if V has a local maximum? Explain.

(b) Determine the type and stability of the equilibria of the equation

$$\ddot{x} = -4x(1 - x^2).$$

(c) Determine the type and stability of $x = \dot{x} = 0$ for the equation

$$\ddot{x} = -4x(1 - x^2) - x^2\, \dot{x}.$$

(4) Let

$$B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, P = \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}, \text{ and } A = P^{-1}BP.$$

(a) Compute e^{tB}. Justify.

(b) Find a periodic orbit for the differential equation $\dot{x} = Bx$.

(c) Find a periodic orbit for the differential equation $\dot{x} = Ax$.