Department of Mathematical Sciences
George Mason University

August, 2011

Ordinary Differential Equations - Preliminary Exam.

Closed books, closed notes. Show work for full credit. No calculators allowed.

(1) Consider a system of equations of the form

$$\dot{x} = Ax,$$

with $x \in \mathbb{R}^2$.

(a) Solve the system directly by directly computing the exponential in the case of

$$A_1 = \begin{pmatrix} 6 & -9 \\ 4 & -6 \end{pmatrix}.$$

That is, use the definition of the exponential of a matrix in order to compute the solution. Is the equilibrium at the origin stable or unstable? Explain.

(b) Solve the system directly in the case of

$$A_2 = \begin{pmatrix} -2 & -6 \\ 1 & 3 \end{pmatrix}.$$

Again use the definition of the exponential of a matrix to find the solution. Is the equilibrium at the origin stable or unstable? Explain.

(c) Determine whether the equilibrium at the origin is stable or unstable in the case of

$$A_3 = \begin{pmatrix} -2 & -6 \\ 0 & -3 \end{pmatrix}.$$

Explain.

(2) (a) Let

$$V(x) = -\int_{x_0}^{x} F(s)\, ds.$$

Show that the energy

$$E = \frac{m\dot{x}^2}{2} + V(x)$$

is constant along solutions of the differential equation $m\ddot{x} = F(x)$.

(b) For the Duffing equation

$$\ddot{x} = -x - x^3,$$

write the equation as a system of equations. Find all equilibria for the system. Describe the stability and the type of each equilibrium solution. (Stability is one of: unstable, stable, or asymptotically stable. Type is one of: Saddle, node, focus, center, center focus, or other.) Justify your answer.

(c) For the damped Duffing equation

$$\ddot{x} = -x - x^3 - \dot{x},$$

write the equation as a system of equations. Determine the type and stability of the equilibrium solution at $x = \dot{x} = 0$. Justify your answer.

(3) Consider the system

$$\dot{x} = x - 2y - x(x^2 + 3y^2)$$
$$\dot{y} = 2x + y - y(x^2 + 3y^2)$$

Changing to polar coordinates gives the system

$$\begin{aligned} \dot{r} &= r - r^3(1 + 2\sin^2\theta) \\ \dot{\theta} &= 2. \end{aligned}$$

(You do not need to verify this.)

(a) Find all equilibria for this system.

(b) Find the maximum radius r_1 such that all solutions are crossing outward across it. Find the minimum radius r_2 such that all solutions are crossing inward across it.

(c) Prove that there is a periodic orbit somewhere in the annulus $r_1 \le r \le r_2$.

(4) Consider the system

$$\begin{aligned} \dot{x} &= \mu + x^2 \\ \dot{y} &= -y \\ \dot{z} &= z. \end{aligned}$$

(a) For all values of μ, find and classify all equilibria.

(b) For all values of μ such that there are equilibria, describe the global stable and unstable manifolds of the equilibria.

(c) For which μ value is there a bifurcation? What type of bifurcation is it? Draw a bifurcation diagram.