Ordinary Differential Equations - Preliminary Exam.

Closed books, closed notes. Show work for full credit. No calculators allowed.

(1) Consider a linear system of the form \( \dot{x} = Ax \), with \( x \in \mathbb{R}^2 \).

(a) Solve the system and find stable and unstable manifolds in the case of

\[
A = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}
\]

(b) Sketch the phase portrait of the above system, and classify its equilibrium.

(c) Show that in general the only invariant lines for this system are the lines \( ax_1 + bx_2 = 0 \), where \( v = (-b, a)^T \) is an eigenvector of \( A \).

(2) Suppose that \( A \) is an \( n \times n \)-matrix.

(a) If \( A^2 = 0 \), find an explicit formula for \( e^{tA} \).

(b) If \( A^2 = A \), show that \( A^k = A \) for all \( k \geq 2 \) and use this to find an explicit formula for \( e^{tA} \).

(c) Determine the stability of the origin of the linear system \( \dot{x} = Ax \) if the coefficient matrix satisfies \( A^2 = 0 \). What can you say if \( A^2 = A \)?

(3) Consider the planar differential equation

\[
\begin{align*}
\dot{x} &= y \\
\dot{y} &= -x + x^3 - y^3
\end{align*}
\]

(a) Find all equilibrium solutions of this system. For each equilibrium solution, what conclusions about their stability can be drawn from a linear stability analysis?

(b) Show that the equilibrium \((0,0)\) is asymptotically stable. (Hint: Try using a Lyapunov function of the form \( V(x, y) = x^2 - \frac{x^4}{2} + y^2 \).

(4) Let \( f : \mathbb{R} \to \mathbb{R} \) be continuously differentiable with \( f(x_0) = 0 \) for some \( x_0 \in \mathbb{R} \), and consider the nonlinear oscillator equation \( \ddot{x} + f(x) = 0 \) in the form

\[
\begin{align*}
\dot{x} &= y \\
\dot{y} &= -f(x)
\end{align*}
\]

(a) Show that in this system, the equilibrium \((x_0, 0)\) is not attracting. Give an example in which the equilibrium is stable. (Hint: Use the fact that this system is Hamiltonian.)

(b) Which condition on \( f'(x_0) \) is necessary and sufficient for the equilibrium \((x_0, 0)\) to be hyperbolic? In the hyperbolic case, sketch the local form of the stable and unstable manifolds via their tangent spaces.