Department of Mathematical Sciences
George Mason University

January, 2011

# Ordinary Differential Equations - Preliminary Exam.

Closed books, closed notes. Show work for full credit. No calculators allowed.

(1) Consider a linear system of the form $\dot{x} = Ax$, with $x \in \mathbb{R}^2$.

(a) Solve the system and find stable and unstable manifolds in the case of

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$$

(b) Sketch the phase portrait of the above system, and classify its equilibrium.

(c) Show that in general the only invariant lines for this system are the lines $ax_1 + bx_2 = 0$, where $v = (-b, a)^T$ is an eigenvector of $A$.

(2) Suppose that $A$ is an $n \times n$-matrix.

(a) If $A^2 = 0$, find an explicit formula for $e^{tA}$.

(b) If $A^2 = A$, show that $A^k = A$ for all $k \geq 2$ and use this to find an explicit formula for $e^{tA}$.

(c) Determine the stability of the origin of the linear system $\dot{x} = Ax$ if the coefficient matrix satisfies $A^2 = 0$. What can you say if $A^2 = A$?

(3) Consider the planar differential equation

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x + x^3 - y^3 \end{aligned}$$

(a) Find all equilibrium solutions of this system. For each equilibrium solution, what conclusions about their stability can be drawn from a linear stability analysis?

(b) Show that the equilibrium $(0, 0)$ is asymptotically stable. (Hint: Try using a Lyapunov function of the form $V(x, y) = x^2 - \frac{x^4}{2} + y^2$.)

(4) Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable with $f(x_0) = 0$ for some $x_0 \in \mathbb{R}$, and consider the nonlinear oscillator equation $\ddot{x} + f(x) = 0$ in the form

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -f(x) \end{aligned}$$

(a) Show that in this system, the equilibrium $(x_0, 0)$ is not attracting. Give an example in which the equilibrium is stable. (Hint: Use the fact that this system is Hamiltonian.)

(b) Which condition on $f'(x_0)$ is necessary and sufficient for the equilibrium $(x_0, 0)$ to be hyperbolic? In the hyperbolic case, sketch the local form of the stable and unstable manifolds via their tangent spaces.