

Ordinary Differential Equations Preliminary Exam

This exam consists of 4 questions.

(1) Suppose that A is an $n \times n$ -matrix such that $A^2 = I$.

- (a) Find an explicit formula for e^{tA} .
- (b) What can you say about the stability properties of the origin of the linear system $\dot{x} = Ax$? Are there nontrivial stable and/or unstable subspaces? Give a general answer, and then illustrate your answer using the matrix

$$A = \begin{bmatrix} 2 & -3 & 4 & -4 \\ 1 & -2 & 4 & -4 \\ 0 & 0 & 2 & -3 \\ 0 & 0 & 1 & -2 \end{bmatrix}.$$

(2) Assume that A and P are two by two matrices such that P is invertible, and $B = P^{-1}AP$, where

$$B = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}$$

Let $x, y \in \mathbb{R}^2$.

- (a) Find a periodic orbit for the differential equation $\dot{x} = Bx$.
- (b) Find a periodic orbit for the differential equation $\dot{y} = Ay$.

(3) For the equation

$$\ddot{x} - x + x^3 = 0,$$

- (a) Write the equation as a first order system of differential equations.
- (b) Find a Hamiltonian for the system.
- (c) Find all fixed points and their stability type.
- (d) Plot the phase portrait.

(4) Consider the planar system

$$\begin{aligned}\dot{x} &= 2x - y - \frac{2x^3 + 2xy^2 - xy}{\sqrt{x^2 + y^2}} \\ \dot{y} &= 2y + x - \frac{2x^2y + 2y^3 + x^2}{\sqrt{x^2 + y^2}}\end{aligned}$$

which in polar coordinates is given by

$$\begin{aligned}\dot{r} &= 2r(1 - r) \\ \dot{\theta} &= 2 \sin^2(\theta/2).\end{aligned}$$

(This does not have to be verified.)

- (a) Find all equilibrium points of this system and determine their stability properties. Are the equilibria asymptotically stable?
 - (b) Sketch the phase portrait of the system, and find all stable and unstable manifolds of the equilibria.
 - (c) Is the above system a gradient system? Justify your answer.
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