Ordinary Differential Equations Preliminary Exam

This exam consists of 4 questions.

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(1) Given a planar system of linear equations of the form $\dot{x} = Ax$, where

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

classify the stability of the equilibrium solution at the origin using the trace and the determinant of A. Specifically, for the trace-determinant plane $\left\{(\tau, \delta) \in \mathbb{R}^2 : \tau = \text{trace}(A), \delta = \det(A)\right\}$, identify the regions where the origin is a:

(a) Saddle

(b) Stable node

(c) Stable focus (also called a stable spiral)

(d) Unstable node

(e) Unstable focus (also called an unstable spiral)

(f) Center

(2) Consider the following planar system given in polar coordinates:

$$\begin{aligned} \dot{r} &= r(1-r) \\ \dot{\theta} &= \sin\theta \end{aligned}$$

where $r \geq 0$ and $\theta \in [0, 2\pi)$.

(a) Draw the phase portrait of the above system in the (x, y)-plane.

(b) Find all equilibrium solutions of the system and determine their stability.

(c) For each equilibrium determine its global stable and unstable manifolds. Justify your answers.

(3) Let $H : \mathbb{R}^2 \to \mathbb{R}$. Recall that a system of the form

$$\begin{aligned} \dot{x} &= \frac{\partial H}{\partial y}(x, y) \\ \dot{y} &= -\frac{\partial H}{\partial x}(x, y) \end{aligned}$$

is called a Hamiltonian system, with Hamiltonian function $H(x, y)$.

(a) Let $(x(t), y(t))$ be a solution to a Hamiltonian system. Show that $\frac{d}{dt}(H(x(t), y(t))) = 0$.

(b) Write the system $\ddot{x} = -x + x^3$ as a first order system of two ordinary differential equations.

(c) Show that there is a Hamiltonian function for this system.

(d) Use this Hamiltonian function to verify the stability type for the equilibrium solution at the origin.

(4) Consider the system

$$\begin{aligned} \dot{x} &= y - x \\ \dot{y} &= rx - y - xz \\ \dot{z} &= xy - z \end{aligned}$$

where r is a real parameter.

(a) Show that for $r > 0$ all solutions of this system are bounded as $t \to \infty$.
(Hint: Verify that the function $V(x, y, z) = rx^2 + y^2 + (z - 2r)^2$ is a Lyapunov function outside the ellipsoid $rx^2 + y^2 + (z - r)^2 = r^2$.)

(b) State the Hartman-Grobman Theorem.

(c) Describe the local dynamics near the origin $(0, 0, 0)$ for both $r = 2$ and $r = -2$ using the Hartman-Grobman theorem.
