Department of Mathematical Sciences George Mason University

Ordinary Differential Equations Preliminary Exam

This exam consists of 4 questions.

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(1) Consider the linear planar system $\dot{x} = Ax$ for the parameter-dependent matrix

$$A = \begin{bmatrix} \alpha & 1 \\ 1 & \alpha \end{bmatrix} , \qquad \alpha \in \mathbb{R} .$$

(a) For which values of α is the origin of this system stable, for which values is it a saddle?

(b) For the case $\alpha = -5$, let $(x_1(t), x_2(t))$ be a nontrivial solution of the above linear system with the initial value $(x_1(0), x_2(0)) = (a, b) \neq (0, 0)$. Find all possible values of $\lim_{t\to\infty} x_2(t)/x_1(t)$. Which initial conditions (a, b) lead to which limit?

(2) Let $g : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function, and consider the planar system

$$\begin{aligned} \dot{x} &= -yg(x^2 + y^2) \\ \dot{y} &= xg(x^2 + y^2) . \end{aligned}$$

(a) Transform the system into polar coordinates.

(b) If the function g has no zero, sketch all possible phase portraits.

(c) For $g(\rho) = \rho - 1$, sketch the phase portrait, find all possible equilibria, and determine their stability.

(3) Consider the planar system

$$\begin{aligned} \dot{x} &= 10 - x - \frac{4xy}{1 + x^2} \\ \dot{y} &= 3x \left(1 - \frac{y}{1 + x^2}\right) . \end{aligned}$$

(a) Find the unique equilibrium solution and show that it is unique.

(b) Show that the rectangle $R = \{(x, y) \in \mathbb{R}^2 \ : \ 0 < x < 10 \ , \ 0 < y < 101\}$ is positively invariant. (You may use without proof that solutions cannot escape through the four corners of R.)

(c) Using the fact that the equilibrium from part (a) is repelling (you do not have to verify this), show that there is a periodic orbit in R.

(4) Consider the planar system

$$\begin{aligned} \dot{x} &= -x + 2xy^2 \\ \dot{y} &= xy + x^2 y . \end{aligned}$$

(a) Find all equilibrium solutions. Which invariant manifolds exist for the equilibria on the y-axis?

(b) Describe the phase portrait in a neighborhood of $(0, 0)$ as precisely as possible. In particular, what are the possible ω-limit sets of orbits starting near $(0, 0)$? (Hint: Linearize about $(0, 0)$, and use the generalized Hartman-Grobman Theorem.)
