Ordinary Differential Equations Preliminary Exam

This exam consists of 4 questions.

(1) Consider the linear planar system $\dot{x} = Ax$ for the parameter-dependent matrix

$$A = \begin{bmatrix} \alpha & 1 \\ 1 & \alpha \end{bmatrix}, \quad \alpha \in \mathbb{R}.$$ 

(a) For which values of $\alpha$ is the origin of this system stable, for which values is it a saddle?

(b) For the case $\alpha = -5$, let $(x_1(t), x_2(t))$ be a nontrivial solution of the above linear system with the initial value $(x_1(0), x_2(0)) = (a, b) \neq (0, 0)$. Find all possible values of $\lim_{t \to \infty} x_2(t)/x_1(t)$. Which initial conditions $(a, b)$ lead to which limit?

(2) Let $g : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function, and consider the planar system

$$\dot{x} = -yg(x^2 + y^2)$$
$$\dot{y} = xg(x^2 + y^2).$$

(a) Transform the system into polar coordinates.

(b) If the function $g$ has no zero, sketch all possible phase portraits.

(c) For $g(\rho) = \rho - 1$, sketch the phase portrait, find all possible equilibria, and determine their stability.

(3) Consider the planar system

$$\dot{x} = 10 - x - \frac{4xy}{1 + x^2}$$
$$\dot{y} = 3x \left( 1 - \frac{y}{1 + x^2} \right).$$

(a) Find the unique equilibrium solution and show that it is unique.

(b) Show that the rectangle $R = \{(x, y) \in \mathbb{R}^2 : 0 < x < 10, 0 < y < 101\}$ is positively invariant. (You may use without proof that solutions cannot escape through the four corners of $R$.)

(c) Using the fact that the equilibrium from part (a) is repelling (you do not have to verify this), show that there is a periodic orbit in $R$.

(4) Consider the planar system

$$\dot{x} = -x + 2xy^2$$
$$\dot{y} = xy + x^2y.$$ 

(a) Find all equilibrium solutions. Which invariant manifolds exist for the equilibria on the $y$-axis?

(b) Describe the phase portrait in a neighborhood of $(0, 0)$ as precisely as possible. In particular, what are the possible $\omega$-limit sets of orbits starting near $(0, 0)$? (Hint: Linearize about $(0, 0)$, and use the generalized Hartman-Grobman Theorem.)