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# Numerical Analysis - Preliminary Exam.

Please do FIVE of the following seven problems.
Closed books, closed notes. Show work for full credit. No calculators allowed.

**Problem 1** (a) Use the Newton-Raphson method: $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$ to establish the following iterative formula for finding $\sqrt{9}$: $x_{n+1} = \dfrac{x_n^2 + 9}{2x_n}, n > 0$.

(b) Show that if the sequence $x_n$ converges to $x^* = 3$, the convergence rate is quadratic.

**Problem 2** Provide a rigorous proof of the fact that if Gaussian elimination with partial pivoting is applied to a matrix $A$ that is diagonally dominant by columns, then no row interchanges will occur.

**Problem 3** (a) Consider the formula $\displaystyle\int_0^h f(x)\ dx = h\left\{Af(0) + Bf\left(\frac{h}{3}\right) + Cf(h)\right\}$. Find $A, B, C$ such that this formula is exact for all polynomials of degree less than or equal to 2.

(b) Suppose that the Trapezoidal rule $\displaystyle\int_0^h f(x)\ dx = \frac{h}{2}\{f(0) + f(h)\}$ applied to $\displaystyle\int_0^2 f(x)\ dx$ gives the value $\dfrac{1}{2}$ while the quadrature rule in part (a) applied to the same integral gives the value $\dfrac{1}{4}$. If $f(0) = 3$, then show that this implies that $f\left(\dfrac{2}{3}\right) = 1$.

**Problem 4** Use the linear ODE $x' = \lambda x$ to analyze accuracy and stability of the Heun's method $x_{k+1} = x_k + \frac{h_k}{2}(k_1 + k_2)$, where $k_1 = f(t_k, x_k), k_2 = f(t_k + h_k, y_k + h_k k_1)$. Prove that the method is 2nd order accurate and characterize its stability region.

**Problem 5** Suppose that the altitude of the trajectory of a projectile is described by the 2nd order ODE $u'' = -4$. Suppose that the projectile is fired from position $t = 0$ and height $u(0) = 1$ and is to strike a target at position $t = 1$, also of height $u(1) = 1$.

(a) Solve this BVP by the shooting method: use the trapezoid rule with step $h = 1$ to derive a system of 2 equations to determine the initial slope at $t = 0$ required to hit the desired target at $t = 1$. Using the initial slope found above and a step size $h = 0.5$, estimate the projectile height at $t = 0.5$.

(b) Solve the same BVP using a finite difference method with $h = 0.5$. Compare the height estimation at $t = 0.5$ with that obtained via shooting method.

**Problem 6** Consider solving a linear system $Ax = b$ using an iterative method

$$\mathbf{x}^{(k+1)} = B\mathbf{x}^{(k)} + \mathbf{c}$$

for $k \geq 0$ with a given initial approximation $\mathbf{x}^{(0)}$.

(a) Show that if the iterative method is convergent to $\mathbf{x}^*$ for all initial choices $\mathbf{x}^{(0)}$, then the spectral radius of matrix $B$ has to satisfy $\rho(B) < 1$.

(b) Show that if $||B|| < 1$ then

$$||\mathbf{x}^{(k)} - \mathbf{x}^*|| \leq \frac{||B||}{1 - ||B||} ||\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)}||.$$

**Problem 7** (a) State the definition of an orthogonal matrix

(b) Find an orthogonal matrix $Q$ and a number $\alpha$ such that

$$Q \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \alpha \\ 0 \end{bmatrix}$$