Numerical Analysis Preliminary Examination questions
January 2018

Instructions: Closed book. NO CALCULATORS are allowed. This examination contains **seven** problems, each worth 20 points. Do any **five** of the seven problems. Show your work. Clearly indicate which **five** are to be graded.

PLEASE GRADE PROBLEMS: 1 2 3 4 5 6 7

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1. Consider the Newton-Raphson method with given initial guess x_0 used to generate a sequence $\{x_n\}, n > 0$ to calculate the reciprocal of the number N by solving $\frac{1}{x} = N$.

   (a) Show that the iteration can be written as $n \geq 0$, $x_{n+1} = x_n(2 - Nx_n)$.

   (b) Show that if the sequence $\{x_n\}$ converges to the root, then the convergence rate is *quadratic*.

2. (a) Given n distinct data values (t_i, y_i), $i = 1, ..., n$, $t_i \in \mathbb{R}^1$, $y_i \in \mathbb{R}^1$, $t_1 < t_2 < \cdots < t_n$, show that there exists a unique polynomial of degree at most $n-1$ that interpolates this data. Describe a method to find this polynomial.

   (b) Let $P_1(f;t)$ be a polynomial of degree 1 interpolating the function $f(t)$ at t_1, t_2. If $f \in C^2[t_1, t_2]$, show that the error in linear interpolation satisfies,

$$f(x) - P_1(f;t) = (t-t_1)(t-t_2)\frac{f''(\xi(t))}{2}, \qquad t_1 < t < t_2$$

3. Consider solving the linear system of equations $Ax = b$ for an invertible matrix A and a fixed right hand side $b \neq 0$.

   (a) Given an approximate solution $\hat{x} \neq 0$, set $\hat{b} = A\hat{x}$ and define the relative forward error (RFE) and relative backward error (RBE) for this problem.

   (b) Find a formula for the condition number of the problem by taking the supremum of the ratio between the RFE and RBE over all inputs $\hat{b}$ such that $\hat{b} \neq b$. (Recall that $||A|| = \sup_{v \neq 0} \frac{||Av||}{||v||}$)

   (c) Compare the condition number of the above problem to the condition number of the matrix A and explain the difference.

4. (a) Derive the Gaussian quadrature rule using two nodes for approximating the integral $\int_{-1}^{1} f(x)x^n\,dx$ where $n \in \mathbb{N}$ is odd.

   (b) Apply your quadrature rule to $f(x) = x^4$ and compare to the true value of the integral. Show that the error is $\mathcal{O}(n^{-3})$.

5. Assume that f satisfies $|f(x) - f(y)| < L|x-y|^\alpha$ for some $L, \alpha > 0$ and for all $x, y \in \mathbb{R}$.

   (a) Show that when $\alpha \neq 1$ we have $|f^n(x) - f^n(y)| < c\left(\frac{|x-y|}{c}\right)^{\alpha^n}$ where $c = L^{\frac{1}{1-\alpha}}$.

   (b) Assume that $f(\hat{x}) = \hat{x}$. In terms of L and α describe the set of x such that fixed point iteration is guaranteed to converge to $\hat{x}$. Consider $\alpha < 1, \alpha = 1$, and $\alpha > 1$.

6. Consider solving the matrix system $Ux = y$ where $U = (u_{ij})$ is a $n \times n$ upper triangular matrix by the following back-substitution algorithm:

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for $i = n : -1 : 1$

  for $j = i + 1 : n$

    $y_i = y_i - u_{ij} * x_j$

  end

  $x_i = \dfrac{y_i}{u_{ii}}$

end

Calculate the number of multiplications and divisions required to perform the above algorithm.

7. Consider the Initial Value Problem, $y'(t) = f(t, y(t)),\ a \leq t \leq b,\ y(a) = y_0$. Let f satisfy a Lipschitz condition in y given by $|f(t,x) - f(t,y)| \leq L|x-y|$, where L is a constant. For $i = 0, 1, \ldots, n$, let the solution $y(t_i)$ and its approximation y_i generated by *Euler's method* satisfy respectively $y(t_{i+1}) = y(t_i) + hy'(t_i) + \frac{h^2}{2}y''(\xi_i)$ and $y_{i+1} = y_i + hf(t_i, y_i)$, where ξ_i is some point in the interval $[t_i, t_{i+1}]$, $a = t_0, t_1, \ldots, t_n = b$ is a discrete set of equally spaced points with $t_i = a + ih$ and stepsize $h = \dfrac{b-a}{n}$. Moreover, if $y''(t)$ exists and is continuous for $a \leq t \leq b$ and that $|y''(t)| \leq M$, then prove that for each $i = 0, 1, \ldots, n$:

$$|y_i - y(t_i)| \leq \frac{Mh}{2L}\left[e^{L(b-a)} - 1\right].$$