Numerical Analysis. Preliminary Exam January 2016.

Closed book. Closed notes. No phones or calculators.

Solve any 5 problems. Circle the numbers of 5 problems to be graded.

1 2 3 4 5 6 7

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1. Which method of solving a linear system of equations quicker in general for large systems: *LU* factorization or Gauss-Jordan elimination? Justify your answer by calculation the dominant terms in the arithmetic operations count for both methods.

2. Let $\mathbf{b} \in \mathbb{R}^n$ be given and $\mathbf{A} \in \mathbb{R}^{n\times n}$ be a strictly diagonally dominant matrix, namely,

$$\max_{1\le i\le n} \sum_{j\ne i} \frac{|a_{ij}|}{|a_{ii}|} = r < 1.$$

   (a) Show that the *Jacobi* iteration $\mathbf{x}_{k+1} = \mathbf{D}^{-1}(\mathbf{L}+\mathbf{U})\mathbf{x}_k + \mathbf{D}^{-1}\mathbf{b}$ converges regardless of the initial guess $\mathbf{x}_0 \in \mathbb{R}^n$.

   (b) Show that the *Gauss-Seidel* iteration $\mathbf{x}_{k+1} = (\mathbf{D}-\mathbf{L})^{-1}\mathbf{U}\mathbf{x}_k + (\mathbf{D}-\mathbf{L})^{-1}\mathbf{b}$ converges regardless of the initial guess $\mathbf{x}_0 \in \mathbb{R}^n$.

3. Suppose that a function g maps the interval $[a, b]$ into itself and g satisfies a Lipschitz condition with a Lipshitz constant $0 \le \lambda < 1$, then

   (a) Show that the sequence $x_{n+1} = g(x_n), n = 0, 1, \ldots$ (for any initial approximation $x_0 \in [a, b]$) converges to a unique fixed point ξ in $[a, b]$.

   (b) Moreover, prove the following error bound involved in using the sequence x_n to approximate ξ given by:

$$|x_n - \xi| \le \frac{\lambda^n}{1-\lambda}|x_1 - x_0| \quad \forall n \ge 1.$$

4. Prove that any local minimum of a convex function f on a convex set $S \subseteq \Re^n$ is a global minimum of f on S.

5. If the integrand f is a twice continuosly differentiable convex function on $[a, b]$, show that the composite midpoint and and trapezoid quadrature rules satisfy the following properties for any $k = 1, 2, \ldots$:

$$M_k(f) \le \int_a^b f(x)dx \le T_k(f).$$

6. Consider the initial-value problem $y'(t) = f(t, y)$ for $0 \le t \le 1$ with $y(0) = a$. Consider the one-step method

$$y_{k+1} = y_k + h\ \Phi(t_k, y_k, h)$$

with $y_0 = a,\ h = 1/N$, and $t_k = kh$ for $k = 0, 1, \cdots, N$. Assume that there is a constant L such that $|\Phi(t, y, h) - \Phi(t, z, h)| \le L\ |y - z|$ for all $t, y, z \in \mathcal{R}$. Furthermore, assume that the solution $y(t)$ satisfies $|y(t + h) - y(t) - h\Phi(t, y(t), h)| \le c\ h^{p+1}$ for all $t, h \in [0, 1]$. Prove that

$$|y_N - y(1)| \le c\ \frac{h^p}{L}\ (e^L - 1).$$

7. Consider the two-point boundary value problem with parameter $b \in \mathbb{R}$:

$$-u'' + bu' + u = 2x \quad \text{in } (0, 1), \quad \text{with } u(0) = u(1) = 0. \tag{1}$$

   (a) Write the finite difference approximation of (1) using *centered* differences and *upwind* differences on a uniform mesh with meshsize h. Write the matrix of the system and examine how the equations would change if $u(1) = \alpha \ne 0$.

   (b) Set up the finite element approximation for this problem, based on piecewise linear elements in equidistant points. State the convergence rate in an appropriate norm.