Numerical Analysis Preliminary Examination questions
January 2019

Instructions: Closed book. NO CALCULATORS are allowed. This examination contains **seven** problems, each worth 20 points. Do any **five** of the seven problems. Show your work. Clearly indicate which **five** are to be graded.

PLEASE GRADE PROBLEMS: 1 2 3 4 5 6 7

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1. Assume that $fl(x)$ is a floating point representation of a real number x with relative error bounded above by $\delta > 0$.

   (a) Show that $\left|\frac{\log(1-\delta)}{\log x}\right|$ is an upper bound on the relative error of $\log(fl(x))$.

   (b) Find the condition number of $\log x$ and explain the connection to part (a).

2. Show that for $n \geq 0$, the sequence

$$x_{n+1} = \frac{1}{2}x_n\left(1 + \frac{a}{x_n^2}\right)$$

   has convergence of second order to the root $\sqrt{a}$.

3. Solve the following system using Cholesky factorization:

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 29 & 41 \\ 3 & 41 & 179 \end{pmatrix} \vec{x} = \begin{pmatrix} 40 \\ 510 \\ 1690 \end{pmatrix}$$

4. Gauss Quadrature

   (a) Derive the two point Gaussian integration formula for

$$I(f) = \int_0^1 f(x)\,dx.$$

   (b) Evaluate the integral using the two-point Gauss Quadrature formula derived:

$$\int_0^1 \frac{dx}{5x+1}.$$

5. Consider the two-point boundary value problem:

$$-u'' + \exp(x)u = \tan(x) \quad \text{in } (0,1), \quad \text{with } u(0) = u(1) = 0. \tag{1}$$

(a) Write the finite difference approximation of (1) using *centered* differences on a uniform mesh with meshsize h. Write the matrix of the system and examine how the equations would change if $u(1) = \alpha \neq 0$.

(b) Set up the finite element approximation for this problem, based on piecewise linear elements in equidistant points. State the convergence rate in an appropriate norm.

6. Consider the matrix

$$\begin{pmatrix} \sqrt{3} & 0 & -\sqrt{3} \\ 0 & a & 0 \\ \sqrt{3} & 0 & \sqrt{3} \end{pmatrix}$$

Explain why normalized power iteration fails to converge for almost every initial vector in $\mathbb{R}^3$ when $a = 1$. For what values of a will normalized power iteration converge for almost every vector in $\mathbb{R}^3$? What will it converge to?

7. Let h be the step size used for interpolating the function $f(x) = \sin(x)$ in $\left[0, \frac{\pi}{4}\right]$ and let $p(x)$ be the quadratic (three nodes) Lagrange interpolating polynomial using equally spaced nodes. Show that $|f(x) - p(x)| \leq \frac{h^3}{9\sqrt{3}}$.