Numerical Analysis Preliminary Examination questions
January 2017

__Instructions__: NO CALCULATORS are allowed. This examination contains **seven** problems, each worth 20 points. Do any **five** of the seven problems. Show your work. Clearly indicate which **five** are to be graded.

PLEASE GRADE PROBLEMS: 1 2 3 4 5 6 7

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1. Assume that $fl(x)$ is a floating point representation of a real number x with relative error bounded above by δ. Show that $\delta\frac{|x|+|y|}{|x+y|}$ is an upper bound on the relative error of $fl(x) + fl(y)$ (relative to $x + y$) and use this to explain the problem of cancellation.

2. Consider the boundary value problem (BVP) $y'' = 3yy'$ with $y(0) = 0$ and $y(1) = 2/3$.

   (a) Rewrite the ODE as a two dimensional first order ODE.

   (b) Set up a a forward Euler integrator with 3 grid points $0, 0.5$, and 1.

   (c) Find the estimated value of $y(1)$ for the initial guesses $y'(0) = 0$ and $y'(0) = 1$.

   (d) Compute two iterations of the bisection method to improve your estimate of $y'(0)$.

   (e) What is this type of BVP solver called?

3. Consider *Steffensen's method* for solving $f(x) = 0$,

$$x_{k+1} = x_k - \frac{f(x_k)}{\varphi(x_k)}, \qquad \varphi(x_k) = \frac{f(x_k + f(x_k)) - f(x_k)}{f(x_k)}$$

   (a) Show that if $f(x) = 0$ and $f'(x) \neq 1$ then $\lim_{y\to x} \frac{f(y)}{\varphi(y)} = 0$, conclude that x is a fixed point of the iteration.

   (b) Show that $\varphi(y) = f'(y) + \frac{1}{2}f''(\xi)f(y)$ for some value of ξ between y and $y + f(y)$.

   (c) What are the advantages and disadvantages of Steffensen's method compared to Newton's method?

4. A matrix $A = (a_{ij})$ of size $n \times n$ is said to be skew-symmetric if $A^T = -A$. Prove the following properties of a skew-symmetric matrix.

   (a) $a_{ii} = 0$ for $i = 1, \ldots, n$.

   (b) $I - A$ is non-singular, where I is the $n \times n$ identity matrix.

5. Let $P(x)$ be the Hermite polynomial interpolating f at the (simple) point $x = 0$ and double point $x = 2$; i.e. $P(0) = f(0)$, $P(2) = f(2)$ and $P'(2) = f'(2)$.

   (a) Show that $\displaystyle\int_0^\infty e^{-x}\ P(x)\ dx = \frac{1}{2}\left(f(0) + f(2)\right).$

   (b) Consider the quadrature formula of the type

   $$\int_0^\infty e^{-x} f(x)\ dx = af(0) + bf(c)$$

   Find a, b and c such that the formula is exact for polynomials of the highest degree possible. (Note that $\displaystyle\int_0^\infty e^{-x}\ x^n\ dx = n!$).

   (c) Compare your result in part (b) with the result in part (a).

6. Let $f : \mathbb{R} \to \mathbb{R}$ be a three times differentiable function.

   (a) What is the order accuracy of the forward difference formula?

   $$f_f'(x) \approx \frac{f(x+h) - f(x)}{h}$$

   (b) What is the order accuracy of the backward difference formula?

   $$f_b'(x) \approx \frac{f(x) - f(x-h)}{h}$$

   (c) What is the order accuracy of the average of f_f' and f_b'?

   Support your answers with an error analysis.

7. (a) Give a criterion for stability and asymptotic stability of the solution to the kth order scalar homogeneous constant-coefficients ODE

   $$y^{(k)} + c_{k-1}y^{(k-1)} + \cdots + c_1 y' + c_0 y = 0.$$

   (b) Suggest a numerical scheme to solve an initial value problem with the above equation.