Numerical Analysis Preliminary Examination questions
January 2013

Instructions: NO CALCULATORS are allowed. This examination contains **seven** problems, each worth 20 points. Do any **five** of the seven problems. Show your work. Clearly indicate which **five** are to be graded.

PLEASE GRADE PROBLEMS: 1 2 3 4 5 6 7

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1. Consider the fixed point iteration method $x_{k+1} = g(x_k), k = 0, 1, \ldots$ for solving the nonlinear equation $f(x) = 0$. Consider choosing an iteration function of the form

$$g(x) = x - af(x) - b(f(x))^2$$

where a, b are parameters. Let z be a root of the $f(x)$. Show that $a = \dfrac{1}{f'(z)}$ and $b = -\dfrac{1}{2}\dfrac{f''(z)}{(f'(z))^3}$ for the iteration method to be of third order.

2. Consider the quadrature formula,

$$\int_a^b f(x)\, dx = h\left\{af(0) + bf\left(\frac{h}{3}\right) + cf(h)\right\}.$$

   (a) Determine a, b and c such that the quadrature formula is exact for polynomials of as high order as possible.

   (b) If the truncation error of the formula is given by $\dfrac{C}{3!}f'''(\xi)$ where $0 < \xi < h$, then show that $C = -\dfrac{h^4}{36}$.

3. Consider approximating the function $f(x) = \sqrt{1+x}$ in the interval $[0, 1]$ at equally spaced nodes $x_j = jh, j = 0, 1, \ldots$. Let the quadratic Lagrange interpolating polynomial to $f(x)$ at x_{j-1}, x_j, x_{j+1} be $P_2(x)$. Show that the error in this approximation satisfies:

$$|f(x) - P_2(x)| \leq \frac{h^3}{24\sqrt{3}}$$

4. Each matrix A and B shown below has determinant equal to ϵ^2 when computed using exact arithmetic,

$$A = \begin{bmatrix} \epsilon & 0 \\ 0 & \epsilon \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1+\epsilon \\ 1-\epsilon & 1 \end{bmatrix},$$

where ϵ is a small positive constant.

Suppose that instead of using exact arithmetic you use a computer with floating point arithmetic to compute the determinant of A and B as ϵ gets smaller and smaller. Assume that your computer has machine precision $\epsilon_{mach} \approx 10^{-16}$, an underflow level, $UFL \approx 10^{-308}$ and an overflow level, $OFL \approx 10^{308}$. At what approximate value of ϵ would you expect each matrix to be numerically singular? Note that your floating point calculation should start with the matrix – not the exact formula ϵ^2 – to compute each determinant. Explain your reasoning for each matrix.

5. Consider the following initial value problem

$$\begin{aligned}\frac{d^2y}{dt^2} &= -y - \sin y, \\ y(t=0) &= y_0, \\ \frac{dy}{dt}(t=0) &= s_0\end{aligned}$$

where y_0 and s_0 are given constants.

(a) Write this as a system of first order differential equations.

(b) Describe a procedure using the Forward Euler method to approximate the solution to this problem at $t = h$ (i.e. apply one time step of Forward Euler with time step h and find an expression for the approximate solution).

(c) Describe a procedure using the Backward Euler method to approximate the solution to this problem at $t = h$ (i.e. apply one time step of Backward Euler with time step h and explain how the approximate solution could be obtained numerically).

(d) Discuss the general advantages and disadvantages of the Forward and Backward Euler methods for solving an initial value problem.

6. (a) Suppose that A is an $n \times n$ matrix. Show that the spectral radius $\rho(A)$ is always less than or equal to any (induced) matrix norm $\|A\|$.

(b) Suppose A is a 5×5 matrix with eigenvalues -5, -4, -1, 0.1, 2.

Given a random starting vector $\boldsymbol{x}_0 \in \mathbb{R}^5$, consider the sequence

$$\frac{\boldsymbol{x}_0}{\|\boldsymbol{x}_0\|}, \frac{A\boldsymbol{x}_0}{\|A\boldsymbol{x}_0\|}, \frac{A^2\boldsymbol{x}_0}{\|A^2\boldsymbol{x}_0\|}, \frac{A^3\boldsymbol{x}_0}{\|A^3\boldsymbol{x}_0\|}, \ldots$$

Show that this sequence converges and identify to what quantity it converges.

7. The QR algorithm for the matrix A is given by

$$A_0 = A = Q_1 R_1 \tag{1}$$

$$A_1 = R_1 Q_1 = Q_2 R_2 \tag{2}$$

$$\ldots \tag{3}$$

$$A_k = R_k Q_k$$

(a) If A is symmetric with eigenvalues $|\lambda_1| > \ldots > |\lambda_n|$, how does one find the eigenvalues and eigenvectors of A using QR algorithm?

(b) Having a QR factorization of the full-rank matrix A ($A = QR$) with $m \geq n$, let $\hat{x}$ be the solution to the least squares problem $\min_x ||b - Ax||$. Show that $||b - A\hat{x}||^2 = ||c_2||^2$, where c_2 is the vector consisting of the last $m - n$ components of Q^*b.

(c) Find an orthogonal matrix G and a number w such that

$$G \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} w \\ 0 \end{bmatrix}$$