Numerical Analysis – Preliminary Exam

Department of Mathematical Sciences

George Mason University

January 2012

Instructions: Answer all four of the following problems. This is a closed book, closed notes exam. Show all of your work for full credit. No calculators or computers are allowed.

1. One iterative method for solving the linear system $A\boldsymbol{x} = \boldsymbol{b}$ [A is $n \times n$, $\boldsymbol{x}$ and $\boldsymbol{b}$ are $n \times 1$] has the form

$$\boldsymbol{x}_{k+1} = G\boldsymbol{x}_k + \boldsymbol{c}, \qquad \text{for } k = 0, 1, 2, \ldots$$

with $G = M^{-1}N$ and $\boldsymbol{c} = M^{-1}\boldsymbol{b}$ where the matrices M and N come from a splitting of the matrix $A = M - N$ with M nonsingular. The iterative scheme is started with an initial guess $\boldsymbol{x}_0$.

(a) Under what conditions on the matrix G will this iterative scheme converge to a solution of $A\boldsymbol{x} = \boldsymbol{b}$? (You may assume that G is a nondefective matrix). Be sure to justify your answer.

(b) The Jacobi method uses $M = D$ and $N = -(L+U)$ where D is a diagonal matrix with the same diagonal elements of A while L and U are strictly lower and upper triangular matrices, respectively so that $A = D + L + U$. The iterative scheme is

$$\boldsymbol{x}_{k+1} \;\; = \;\; -D^{-1}(L+U)\boldsymbol{x}_k + D^{-1}\boldsymbol{b}$$

A modification of this method is the Gauss-Seidel method in which the iterative scheme is given by

$$\boldsymbol{x}_{k+1} \;\; = \;\; -D^{-1}L\boldsymbol{x}_{k+1} - D^{-1}U\boldsymbol{x}_k + D^{-1}\boldsymbol{b}$$

where D, L and U are defined as above. Explain and/or interpret as many similarities and differences between these two methods that you can identify, including implementation details.

2. Consider the overdetermined system $A\boldsymbol{x} = \boldsymbol{b}$ where A is an $m \times n$ matrix of full rank, $\boldsymbol{x}$ is a vector of length n and $\boldsymbol{b}$ is a vector of length m. Here $m > n$. A weighted least squares problem seeks $\boldsymbol{x}$ such that the norm

$$\|D(\boldsymbol{b} - A\boldsymbol{x})\|_2$$

is minimized. Here D is a full rank $m \times m$ diagonal matrix with entries d_{jj} [The case $D = I$ corresponds to the non-weighted least squares problem.]

(a) Write down the corresponding normal equations for the weighted least squares problem in terms of the matrices A and D and the vectors $\boldsymbol{x}$ and $\boldsymbol{b}$.

(b) In the standard (non-weighted) least squares problem another interpretation of the solution $\boldsymbol{x}$ is that it solves the equation $A\boldsymbol{x} = P\boldsymbol{b}$ where P is an orthogonal projector onto the range (or span) of A. Give a formula for the analogous orthogonal projector P_W for the weighted least squares problem.

3. Consider Heun's method for solving ODEs given by $x_{k+1} = x_k + \frac{h_k}{2}(k_1 + k_2)$, where $k_1 = f(t_k, x_k), k_2 = f(t_k + h_k, x_k + h_k k_1)$.

(a) Determine the order of accuracy and the stability region for this method.

(b) Characterize the method as implicit/explicit, single-step or multistep.

(c) Define what it means for the ODE to be considered stiff. Suggest a possible modification of Heun's method to make it more suitable for the solution of stiff equations. Explain.

4. (a) Given the three data points $(-1, 1), (0, 0), (1, 1)$ determine the interpolating polynomial of degree two using monomial basis, Lagrange basis and Newton basis.

(b) Rank the three methods above according to the cost of determining the interpolant, from least to most expensive. Which of the three methods above has the best-conditioned basis matrix?

(c) Derive the formula for the interpolation error $f(x) - p(x)$, where $p(x)$ is a polynomial of degree at most n that interpolates f at $n + 1$ distinct nodes $x_0, x_1, \ldots, x_n$ on the interval $[a, b]$ where f is sufficiently smooth.

(d) Find the interpolation error associated with the interpolants found in (a) on the interval $[-1, 1]$ assuming the function being interpolated is given by $f(x) = x^4$.