Numerical Analysis Preliminary Examination
December 2019

Check here if you are taking this as a preliminary exam: _____

<u>**Instructions**</u>: Closed book. NO CALCULATORS are allowed. This examination contains **seven** problems, each worth 20 points. Do any **five** of the seven problems. Show your work. Clearly indicate which **five** are to be graded.

PLEASE GRADE PROBLEMS: 1 2 3 4 5 6 7

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1. Consider the problem of evaluating a function $f \in C^2(\mathbb{R})$ with a perturbed input $\tilde{x} = x + \delta$.

   (a) Write the formula for the condition number of f.

   (b) Show that the condition number is the same as $\frac{d}{dy} \log f(\exp(y))$ where $y = \log(x)$.

   (c) Consider the absolute forward error in $\log f$ given by $\mathrm{FE} = |\log f(\tilde{x}) - \log f(x)|$, and the absolute backward error in $\log x$ given by $\mathrm{BE} = |\log \tilde{x} - \log x|$. Show that $\lim_{\delta \to 0} \frac{\mathrm{FE}}{\mathrm{BE}}$ is the condition number of f.
   (Recall the expansion $\log(a + b\delta) = \log(a) + \frac{b}{a}\delta + \mathcal{O}(\delta^2)$)

2. Consider Newton's method $x_{k+1} = \phi(x_k)$ where $\phi(x) = x - \frac{f(x)}{f'(x)}$ is the associated fixed point function.

   (a) Turn this into a derivative-free method by replacing the derivative, $f'(x)$, with the forward difference approximation using a fixed h. Give the formula for the new fixed-point function, $\hat{\phi}(x)$.

   (b) Let $x^* = \hat{\phi}(x^*)$ be a fixed point such that $f'(x^*) \neq 0$ and $f \in C^2(\mathbb{R})$, show that

   $$\hat{\phi}'(x^*) = \frac{f(x^*) f''(x^*)}{f'(x^*)} + \mathcal{O}(h).$$

   (c) What is the order of convergence of this method?

3. Derive the one and two point Gaussian quadrature formulas such that

   $$\int_{-1}^{1} f(x) x^4 dx \approx \sum_{j=1}^{n} f(x_j) w_j.$$

   Give bounds on the error of these formulas.

4. (a) Summarize the QR iteration algorithm for finding the eigendecomposition of a matrix A.
   (b) The matrix A is called upper Hessenberg if $A_{ij} = 0$ for all $i > j+1$. Show that if A is upper Hessenberg, this property is preserved by the QR iteration algorithm.

5. (a) Show that for $n+1$ equally spaced nodes $x_i = a + ih$ then for $x \in [a,b]$ we have $|w_n(x)| = \prod_{i=0}^{n} |x - x_i| \le \frac{h^{n+1} n!}{4}$.
   Hint: Assume $x \in [x_{j-1}, x_j]$ and find an upper bound on $(x - x_{j-1})(x_j - x)$ then use a coarse bound on the other terms and note that $j!(n-j+1)! \le n!$ for any j.
   (b) Assume $f \in C^{n+1}[a,b]$, find an upper bound on the interpolation error using $n+1$ equally spaced nodes in terms of n, h, and the derivatives of f.

6. Consider approximating the integral $\int_0^1 g(x)\,dx$ with the composite trapezoid rule $I_n(g)$,

$$I_n(g) = \sum_{k=1}^{n-1} \left(\frac{x_{k+1} - x_k}{2} \right) (g(x_k) + g(x_{k+1}))$$

where $0 = x_0 < x_1 < \cdots < x_n = 1$. Find a piecewise linear function G (so G restricted to $[x_k, x_{k+1}]$ is given by $G(x) = a_k x + b_k$) such that

$$I_n(g) - \int_0^1 g(x)dx = \int_0^1 G(x)g'(x)dx \tag{1}$$

for any $g \in C^1[0,1]$.

Hint. Find a_k, b_k by applying the fundamental theorem of calculus to $\int_{x_k}^{x_{k+1}} (g(x)G(x))'\,dx$ and then expanding the derivative to get an equation that matches (1) on $[x_k, x_{k+1}]$.

7. Consider the boundary value problem (BVP) $-u'' + au' = (1 + \sin(t))u$ for $t \in [0, \pi]$ with boundary conditions $u(0) = b$ and $u'(\pi) = c$.
   (a) Set up a finite difference scheme using centered difference approximations to solve the BVP using $n+1$ grid points $t_i = ih$ with $i = 0, ..., n$ and $h = \pi/n$. Put your system into matrix form.
   (b) Describe and set up another method of solving the BVP (not finite differences).