Numerical Analysis. Preliminary Exam August 2015.

Closed book. Closed notes. No phones or calculators.

Solve any 5 problems. Circle the numbers of 5 problems to be graded.

1 2 3 4 5 6 7

---

1. Let A be $n \times n$ strictly diagonally dominant matrix, i.e. $\sum_{j=1}^{n} |a_{ij}| < |a_{ii}|$ for $i = 1, \ldots, n$. Prove that Jacobi method for solving $Ax = b$ converges from any initial vector.

2. Consider the function $g(x) = e^{-x}$.

   (a) Show that $g(x)$ has an unique fixed point $z \in (-\infty, \infty)$.

   (b) Prove that g is a contraction on $[\ln 1.1, \ln 3]$.

   (c) Prove that $g : [\ln 1.1, \ln 3] \to [\ln 1.1, \ln 3]$

   (d) Prove that $x_{t+1} = g(x_t)$ converges to the unique fixed point $z \in (-\infty, \infty)$ for <u>any</u> $x_0 \in (-\infty, \infty)$.

3. Let $\displaystyle\int_0^1 f(x)\ dx \approx \sum_{i=0}^{N-1} f(x_i)h$ where $x_i = ih$ and $h = \dfrac{1}{N}$. Suppose that $f \in C^1[0,1]$. Prove that $\displaystyle\left|\int_0^1 f(x)\ dx - \sum_{i=0}^{N-1} f(x_i)\ h\right| \leq \frac{h}{2} \max_{0 \leq x \leq 1} |f'(x)|.$

4. Use the first and second order optimality conditions to find all the local constrained minima and maxima of $f(x) = x_1 x_2 x_3$ such that $x_1 + x_2 + x_3 = 3$.

5. (a) Write down a polynomial $P(x)$ of the lowest degree, but not a spline, satisfying $P(1) = P(2) = = P(50) = 10$ and $P(0) = 3$.

   (b) Evaluate $P(-1)$.

6. Let $s(x)$ denote the complete spline of $f(x)$ on the interval $[a,b]$ with knots $a = x_0 < \cdots < x_n = b$; thus $s'(a) = f'(a)$ and $s'(b) = f'(b)$. Set $e(x) = f(x) - s(x)$.

   (a) Integrate twice by parts on each subinterval (x_{i-1}, x_i) to derive the (orthogonality) relation

   $$\int_a^b e''(x)\phi(x)dx = 0,$$

   for all continuous piecewise linear functions ϕ.

   (b) Use (a) to prove the identity (Pythagoras equality)

   $$\int_a^b \left|f''(x)\right|^2 dx = \int_a^b \left|s''(x)\right|^2 dx + \int_a^b \left|f''(x) - s''(x)\right|^2 dx.$$

7. Given the two-point boundary value problem

$$
\begin{aligned}
&-u''(x) + \alpha u(x) = f(x), \quad 0 \le x \le 1, \alpha > 0, \\
&u'(0) = A, \\
&u'(1) = B.
\end{aligned}
$$

(*a*) Set up the finite element approximation for this problem, based on piecewise linear elements in equidistant points. Determine the convergence rate in an appropriate norm.

(*b*) Explain whether $\alpha > 0$ is necessary for the convergence in part (a).