Numerical Analysis Preliminary Examination questions
August 2019

Instructions: Closed book. NO CALCULATORS are allowed. This examination contains **seven** problems, each worth 20 points. Do any **five** of the seven problems. Show your work. Clearly indicate which **five** are to be graded.

PLEASE GRADE PROBLEMS: 1 2 3 4 5 6 7

---

1. Let $x_i^*,\ i \geq 0$ denote positive numbers on a computer. With a unit round-off error δ, $x_i^* = fl(x_i) = x_i\ (1+\epsilon_i)$ with $|\epsilon_i| \leq \delta$, where x_i are positive exact numbers. Consider the expression $S = x_1 \cdot x_2 \cdot x_3 + x_4 \cdot x_5 \cdot x_6$. Let S^* be the floating point approximation of S. Prove that $\dfrac{S^*}{S} \leq e^{6\delta}$.

2. Consider the sequence defined by $x_{n+1} = b+\epsilon\ h(h(x_n))$ for $n = 0, 1, 2, \ldots$, where $x_0 \in \mathbb{R}$ is given and $b, \epsilon \in \mathbb{R}$. Assume that,

   (a) $\exists\ M > 0$ such that, $|h(x)| \leq M$ for all $x \in \mathbb{R}$ and,

   (b) $\exists\ L > 0$ such that, $|h(v) - h(w)| \leq L|v - w|$ for all $v, w \in \mathbb{R}$.

   Prove that if $2ML|\epsilon| < 1$, then there is a unique $z \in \mathbb{R}$ such that $z = b + \epsilon\ h(h(z))$ and the sequence $\{x_n\}_{n=0}^{\infty}$ converges to this value z.

3. Perform a QR factorization of the matrix

$$\begin{pmatrix} 3 & 5 \\ 4 & 6 \end{pmatrix}$$

   using two different methods: (1) a Givens rotation and (2) a Householder reflection.

4. Let A be a $n \times n$ symmetric matrix with distinct eigenvalues and consider the function $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$

$$f(v_1, v_2) = \left(\frac{Av_1}{||Av_1||}, \frac{A\left(v_2 - \frac{\langle v_2, v_1\rangle}{\langle v_1, v_1\rangle} v_1\right)}{\left|\left|A\left(v_2 - \frac{\langle v_2, v_1\rangle}{\langle v_1, v_1\rangle} v_1\right)\right|\right|} \right).$$

   Let $u_1, ..., u_n$ be unit eigenvectors of A with eigenvalues $\lambda_1 > \lambda_2 > \cdots > \lambda_n$.

   (a) Show that (u_i, u_j) is a fixed point of f if $i \neq j$.

   (b) For which pair (u_i, u_j) is this iteration attracting? Explain why.

5. Consider the integral $I(f) = \int_0^1 \sqrt{y} f(y)\, dy$. Let the quadrature approximation of $I(f)$ be given by $Q_M(f) = \sum_{m=1}^M \omega_m f(y_m)$ where $\{(\omega_m, y_m)\}_{m=1}^M$ denotes the quadrature weight-node pairs.

   (a) Let q be a polynomial for which Q_M is exact, i.e., $Q_M(q) = I(q)$. If $\omega_m \geq 0$ then show that
   $$|I(f) - Q_M(f)| \leq \frac{4}{3} \max_{x \in [0,1]} |f(x) - q(x)|.$$
   Hint: Notice that $I(f) - Q_M(f) = I(f - q) + Q_M(q - f)$. Also, check that $\sum_{m=1}^M \omega_m = \int_0^1 \sqrt{y}\, dy = \frac{2}{3}$.

   (b) If f is continuous and $Q_M(f)$ is the Gaussian quadrature with M nodes then use (a) to show that
   $$\lim_{M \to \infty} Q_M(f) = I(f).$$

6. Consider the two-point boundary value problem:
   $$-u'' + u = x \quad \text{in } (0,1), \quad \text{with } u'(0) = u'(1) = 0. \tag{1}$$

   (a) Write the finite difference approximation of (1) using *centered* differences on a uniform mesh with meshsize h. Write the matrix of the system.

   (b) Set up the finite element approximation for this problem, based on piecewise linear elements in equidistant points. State the convergence rate in an appropriate norm.

7. Let $F : D \subset \mathbb{R}^n \to \mathbb{R}^n$ and $\gamma > 0$ satisfy
   $$\|F(x) - F(y)\| \geq \gamma \|x - y\|, \quad \forall x, y \in D$$
   Show that F is invertible and its inverse $F^{-1} : F(D) \to D$ is Lipschitz continuous with constant $1/\gamma$.