Numerical Analysis Preliminary Examination questions
January 2018

<u>**Instructions**</u>: Closed book. NO CALCULATORS are allowed. This examination contains **seven** problems, each worth 20 points. Do any **five** of the seven problems. Show your work. Clearly indicate which **five** are to be graded.

PLEASE GRADE PROBLEMS: 1 2 3 4 5 6 7

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1. Let $f \in C^{1,\alpha}[x_0, x_1]$ be a C^1 function whose derivative f' is Hölder α with $0 < \alpha \leq 1$. Let p be the linear interpolant of f at x_0 and x_1. If $w(t)$ denotes the modulus of continuity of f', show the error estimate

$$|f(x) - p(x)| \leq hw(h) \quad \forall x_0 < x < x_1,$$

   where $h = x_1 - x_0$.
   Hint: Express the differences $f(x) - f(x_0)$ and $f(x_1) - f(x_0)$, which appear when writing $f(x) - p(x)$, in integral form and then combine them. Also recall from the definition of modulus of continuity which implies $|f(x) - f(y)| \leq w(|x - y|)$.

2. Gauss Quadrature

   (a) Derive the two point Gaussian integration formula for

$$I(f) = \int_{-1}^{1} f(x)\, dx.$$

   (b) Use this formula to approximate $I(x^2)$.

3. Let A be diagonalizable, and consider the power iteration sequence $w_{k+1} = Aw_k$ for an initial vector w_0. Find the limit $\lim_{k\to\infty} \frac{w_k}{||w_k||}$ and show how this limit depends on the spectrum of A and the initial vector w_0.

4. Describe an algorithm for finding the inverse of an $N \times N$ matrix which runs in $\mathcal{O}(N^3)$ time and justify the computation complexity.

5. Consider a minimization problem $f(x) = \frac{1}{2}x^T Ax - b^T x$, $x \in \mathbb{R}^n$ and A is symmetric positive definite.

   (a) Prove that $f(x)$ is minimized when $Ax = b$.

   (b) Let x_0 be the starting point. Perform one step of steepest descent method with exact line search.

   (c) Perform one step of both Jacobi and Gauss-Seidel iterative methods to solve $Ax = b$ and compare the two methods.

6. Consider the Newton-Raphson method: $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

   (a) Use the Newton-Raphson method to establish the following iterative formula $x_{n+1} = \dfrac{x_n^2 + 16}{2x_n}$ for $n > 0$ to find the square-root of 16.

   (b) Show that if the sequence $\{x_n\}$ converges to the limit $L = 4$, then the convergence rate is quadratic. (Hint: Consider the error $e_n = x_n - 4$.)

7. Consider the two-point boundary value problem:

$$-u'' + u = \sin(x) \quad \text{in } (0,1), \quad \text{with } u(0) = u(1) = 0. \tag{1}$$

   (a) Write the finite difference approximation of (1) using *centered* differences on a uniform mesh with meshsize h. Write the matrix of the system and examine how the equations would change if $u(0) = \alpha \neq 0$.

   (b) Set up the finite element approximation for this problem, based on piecewise linear elements in equidistant points. State the convergence rate in an appropriate norm.