Numerical Analysis Preliminary Examination questions
August 2017

<u>**Instructions**</u>: Closed book. NO CALCULATORS are allowed. This examination contains **seven** problems, each worth 20 points. Do any **five** of the seven problems. Show your work. Clearly indicate which **five** are to be graded.

PLEASE GRADE PROBLEMS: 1 2 3 4 5 6 7

---

1. Consider a differentiable function $f : \mathbb{R} \to \mathbb{R}$ and let $x^* = fl(x)$ be the floating point approximation of x.

   (a) Show that the relative error in function evaluation is given by:

   $$\left| \frac{f(x) - f(x^*)}{f(x)} \right| \approx \kappa_f(x^*) \left| \frac{x - x^*}{x} \right|$$

   where the *condition number* $\kappa_f(x^*) := \left| \dfrac{x f'(x^*)}{f(x)} \right|$.

   (b) Use the definition of condition number in part (a) to show that the condition number of the product $f(x) \cdot g(x)$ of functions satisfies:

   $$\kappa_{fg}(x) \leq \kappa_f(x) + \kappa_g(x).$$

2. Suppose that a function g maps the interval $[a, b]$ into itself and g satisfies a Lipschitz condition with a Lipshitz constant $0 \leq \lambda < 1$, then

   (a) Show that the sequence $x_{n+1} = g(x_n), n = 0, 1, \ldots$ (for any initial approximation $x_0 \in [a, b]$) converges to a unique fixed point ξ in $[a, b]$.

   (b) Moreover, prove the following error bound involved in using the sequence x_n to approximate ξ given by: $|x_n - \xi| \leq \dfrac{\lambda^n}{1 - \lambda} |x_1 - x_0| \quad \forall n \geq 1$.

3. Prove that Jacobi method for solving the linear system of equations $Ax = b$ with strictly diagonally dominant matrix A converges with a linear rate.

4. (a) Explain how to find the natural cubic spline for n data points $(x_1, y_1), ..., (x_n, y_n)$ such that the resulting linear system $Ax = b$ has a matrix A of size n-by-n.

   (b) What is the largest number of nonzero elements in A?

   (c) Suggest an appropriate numerical method to solve $Ax = b$.

5. (a) Give a formula for a second order approximation of the second derivative $f''(x)$.

   (b) Assume that f is four times continuously differentiable in a neighborhood of x and show that your approximation in part (a) is second order by deriving an appropriate upper bound on the error.

6. Consider applying the forward Euler method to solve the initial value problem, $x' = \lambda x$ with $x(0) = 1$ using a step size $h = t_{k+1} - t_k$.

   (a) Write the formula for $x_{k+1} = x(t_{k+1})$ in terms of $x_k = x(t_k)$.

   (b) For which values of h and λ will this method be stable?

   (c) Repeat steps (a) and (b) for the backward Euler method.

7. (a) Derive the quadrature rule for approximating the integral $\int_a^{a+h} f(x)\,dx$ using the two nodes $x_1 = a$ and $x_2 = a + 2h/3$ such that the quadrature rule is exact for all polynomials up to the highest possible degree.

   (b) Show that the quadrature rule above is exact on all polynomials of degree 2.

   (c) Assume that the node $x_1 = a$ is fixed but you are allowed to change the node x_2. Is $x_2 = a + 2h/3$ the choice for which the quadrature rule is exact for all polynomials up to the highest possible degree? Why or why not?